On abelian subgroups of an infinite 2-group

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In [4] and [7] the following proposition has been proved:

If G is an infinite locally finite group, then G contains an infinite abelian subgroup.

It may be conjectured that the conclusion of the above proposition is still true if one drops the condition of local finiteness. As a contribution to the solution of this problem we prove the following

Theorem.*) *An infinite 2-group contains at least one infinite abelian subgroup.*

Remark. It is a consequence of [2; p. 274, footnote] that not every 2-group is locally finite.

We shall prove our theorem by contradiction. Thus, assume that an infinite 2-group G exists all the abelian subgroups of which are finite. Denote by R the product of all locally nilpotent normal subgroups of G. Application of [3; Corollary (HIRSCH—PLOTKIN), p. 155] yields that R is locally nilpotent. Since for 2-groups local finiteness is equivalent to local nilpotency (cp. [6], Lemma 6, p. 54), R is locally finite.

Consider an abelian subgroup A^* of $G^* = G/R$, and let A be a subgroup of G satisfying $A/R = A^*$. Since A is an extension of a locally finite group by a locally finite group, A is locally finite (cp. [1], Finiteness Principle, p. 166). Application of [7; Lemma, p. 232] yields the finiteness of A. Hence A^* is finite. Thus, every abelian subgroup of G^* is finite. If G were equal to R, then G would be finite by [7; Lemma, p. 232] contradicting our assumption. Hence there exists an element a of order 2 in G^*. Since G^* is a 2-group and $(a \circ x)^{2^{i-1}} = a^{(i)} \circ x$ for all $x \in G^*$ and all integers $i \geqq 1$, a is a left Engel element of G^* (cp. [5], p. 584). Application of [6; Satz, p. 60] yields that $\{a^{G^*}\} \neq 1$ is a locally finite normal subgroup of G^*. Hence there exists a normal subgroup B of G with $B/R = \{a^{G^*}\}$. But B is locally finite and therefore contained in R, contradicting $\{a^{G^*}\} \neq 1$. From that contradiction follows the validity of our theorem.

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*) Dr. O. KEGEL has informed the author that, independently and using a different method, he has obtained the same result.

References

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