On stationary sets and regressive functions

By G. FODOR in Szeged

For every α, ω_α denotes the initial number of $\aleph_\alpha$ and $\aleph_{\mathrm{cf}(\alpha)}$ denotes the least cardinal number m such that $\aleph_\alpha$ can be expressed as the sum of m cardinal numbers $<\aleph_\alpha$. If $\mathrm{cf}(\alpha)=\alpha$ then $\aleph_\alpha$ and ω_α are said to be regular; otherwise they are singular. An ordinal number α is called a limit number if there is no β such that $\alpha=\beta+1$. We say that $\aleph_\alpha$ is a limit cardinal number if α is a limit number. Let now $W(\omega_\alpha)=\{\xi:\xi<\omega_\alpha\}$. We call a subset S of $W(\omega_\alpha)$ confinal to $W(\omega_\alpha)$ if for every $\nu\in W(\omega_\alpha)$ there is a $\mu\in S$ such that $\mu>\nu$. A subset C of $W(\omega_\alpha)$ is called closed if the limit of any fundamental sequence of elements of C belongs to C whenever this limit is smaller than ω_α. Let $M\subset W(\omega_\alpha)$. If $W(\omega_\alpha)-M$ does not contain a closed subset confinal to $W(\omega_\alpha)$ then we say that M is stationary; otherwise it is called non-stationary. We call a function $f(\gamma)$ on $M\subset W(\omega_\alpha)$ into $W(\omega_\alpha)$ regressive if for every $\gamma\in M$ the inequality $f(\gamma)<\gamma$ (and $f(0)=0$ for $0\in M$) holds.

We assume that $\mathrm{cf}(\alpha)>0$ and the set of the regular initial numbers $<\omega_{\mathrm{cf}(\alpha)}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$. We shall prove the following statements.

(i) *Every stationary subset of $W(\omega_\alpha)$ may be expressed as the sum of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets.*

BLOCH [1] has proved this statement for $\alpha=1$.

(ii) *Let S be a stationary subset of $W(\omega_\alpha)$. The set S may be expressed as the sum $\bigcup_{\eta<\omega_{\mathrm{cf}(\alpha)}} S_\eta$ of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets such that for each stationary subset $M\subset S$ there is an ordinal number $\eta_0<\omega_{\mathrm{cf}(\alpha)}$ for which $M\cap S_{\eta_0}$ is a stationary set.*

(iii) *If for every limit number $\xi\in W(\omega_1)$ there exists a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\ldots<f_i(\xi)<\ldots$ converging to ξ then for all but finitely many positive integers i there is a set S_i of the cardinal number $\aleph_1$ such that the set $\{\xi:f_i(\xi)=\gamma\}$ is stationary in $W(\omega_1)$ for each $\gamma\in S_i$.*

This theorem is a generalization of a theorem of B. ROTMAN [4].

By the proof of these statements we shall use the following

Theorem I. *Let ω_α be an initial number which is not confinal to ω, $\{K_\gamma\}_{\gamma<\tau}$ ($\tau\leqq\omega_{\mathrm{cf}(\alpha)}$) a sequence of the type τ of of non-empty and mutually disjoint non-stationary subset of $W(\omega_\alpha)$ and x_γ the first element of K_γ. Let us suppose that the elements x_γ are arranged according to their magnitude, i.e. $x_\gamma<x_\beta$ for $\gamma<\beta$. If the set $\{x_\gamma\}_{\gamma<\tau}$ is non-stationary and in the case $\tau=\omega_{\mathrm{cf}(\alpha)}$ confinal to $W(\omega_\alpha)$ then the set $\bigcup_{\gamma<\tau} K_\gamma$ is non-stationary.* (See [3].)

Theorem II. *Let ω_α be a regular initial number $>\omega$ and ϱ a regular limit ordinal number, $\varrho<\omega_\alpha$. The set of all ordinal numbers $\lambda<\omega_\alpha$ of the second kind which are confinal to ϱ is a stationary subset of $W(\omega_\alpha)$.* (See [2].)

Theorem III. *Let ω_α be an initial number which is not confinal to ω, M a subset of $W(\omega_\alpha)$ and $g(\gamma)$ a regressive function on M. If M is a stationary subset of $W(\omega_\alpha)$ then there exists an ordinal number $\pi<\omega_\alpha$ and a stationary subset N of M such that $g(\gamma)\leqq\pi$ for every $\gamma\in N$.* (See [3].)

First we prove with the method of G. Bloch [4] the following

Lemma 1. *If ω_α is a regular initial number with $\alpha>0$ and ω_β is a given regular initial number smaller than ω_α then every stationary subset of the set of the limit numbers $<\omega_\alpha$ which are confinal to ω_β may be expressed as the sum of $\aleph_\alpha$ mutually disjoint stationary sets.*

Proof. By Theorem II the set A of the limit numbers $<\omega_\alpha$ which are confinal to ω_β is a stationary subset of $W(\omega_\alpha)$. Let us denote by S a stationary subset of the set A.

We prove that there exists a regressive function φ on the set S with the property:

(P) *if $S=R\cup Q$ is a decomposition of S into two disjoint sets R and Q such that φ is bounded on the set R (i.e. there exists an ordinal number $\gamma<\omega_\alpha$ such that $\varphi(\xi)<\gamma$ for every $\xi\in R$), then the set Q is stationary in $W(\omega_\alpha)$.*

Since $S\subset A$ there exists for every element $\xi\in S$ an increasing sequence $\{\xi_\eta\}_{\eta<\omega_\beta}$ of the type ω_β of ordinal numbers $<\xi$ such that $\lim\limits_{\eta<\omega_\beta}\xi_\eta=\xi$. We define now on the set S a sequence $\{f_\eta\}_{\eta<\omega_\beta}$ of the type ω_β of regressive functions as follows: let $\xi\in S$ and

$$f_\eta(\xi)=\begin{cases} 0 & \text{if } \xi=0,\\ \xi_\eta & \text{if is a limit number.}\end{cases}$$

We show now that there exists an ordinal number $\eta<\omega_\beta$ for which the function f_η has the property (P). Suppose on the contrary that $\eta<\omega_\beta$ but f_η does not have the property (P). Then to every ordinal number $\eta<\omega_\beta$ there corresponds a decomposition $S=R_\eta\cup Q_\eta$ of S into two disjoint sets R_η and Q_η such that the function f_η is bounded on the set R_η and the set Q_η is non-stationary in $W(\omega_\alpha)$. Thus by Theorem I the set $\bigcup\limits_{\eta<\omega_\beta}Q_\eta$ is non-stationary in $W(\omega_\alpha)$; consequently the set $W(\omega_\alpha)-\bigcup\limits_{\eta<\omega_\beta}Q_\eta$ contains a closed subset confinal to $W(\omega_\alpha)$. Since

$$W(\omega_\alpha)-\bigcup_{\eta<\omega_\beta}Q_\eta=\big(W(\omega_\alpha)-S\big)\cup\big(\bigcap_{\eta<\omega_\beta}R_\eta\big)$$

and the set $W(\omega_\alpha)-S$ does not contain a closed subset confinal to $W(\omega_\alpha)$, the set $\bigcap\limits_{\eta<\omega_\beta}R_\eta$ is confinal to $W(\omega_\alpha)$. Since for every $\eta<\omega_\beta$ the function f_η is bounded on the set $\bigcap\limits_{\eta<\omega_\beta}R_\eta$ and $\omega_\beta<\omega_\alpha$, there exists an ordinal number $\gamma<\omega_\alpha$ such that each of the functions f_η is bounded by γ, which contradicts the definition of the functions $f_\eta(\eta<\omega_\beta)$. Thus there exists a regressive function φ on the set S with the property (P).

Let H be the set of all ordinal numbers γ for which the set of the solutions ξ of the equation $\varphi(\xi)=\gamma$ is confinal to $W(\omega_\alpha)$. By Theorem III the set is non-empty. For every element $\gamma\in H$ let us denote by M_γ the set of the solutions of the equation $\varphi(\xi)=\gamma$:

$$M_\gamma=\{\xi\in S:\varphi(\xi)=\gamma\}.$$

It is clear that the sets M_γ $(\gamma\in H)$ are mutually disjoint. Put

$$M=\bigcup_{\gamma\in H} M_\gamma.$$

The set H is confinal to $W(\omega_\alpha)$. For if not, the function φ would be bounded on the set M. Consequently, since φ has the property (P), the set $S-M$ would be stationary, which is impossible since by Theorem III there exists a subset S' of $S-M$ which is confinal to $W(\omega_\alpha)$ and an ordinal number $\gamma<\omega_\alpha$ such that $\varphi(\xi)=\gamma$ for every $\xi\in S'$. Thus the set H is confinal to $W(\omega_\alpha)$. Let H' be the set of all elements γ of the set H for which the sets M_γ are non-stationary in $W(\omega_\alpha)$. Put

$$M'=\bigcup_{\gamma\in H'} M_\gamma.$$

Let us denote by x_γ the first element of the set M_γ. By Theorem III the set $\{x_\gamma\}_{\gamma\in H'}$ is non-stationary in $W(\omega_\alpha)$ since for the function

$$g(x_\gamma)=\gamma$$

the relation $g(x_\gamma)\neq g(x_\tau)$ holds if $\gamma\neq\tau$. Thus by Theorem I the set M' is non-stationary in $W(\omega_\alpha)$. Since the function φ has the property (P), it follows that φ is not bounded on the set $S-M$. Consequently the set $H-H'$ is confinal to $W(\omega_\alpha)$. Let γ_0 be the first element of the set $H-H'$ and let

$$L_\gamma=\begin{cases} M_{\gamma_0}\cup(S-\bigcup_{\gamma\in H} M_\gamma)\cup(\bigcup_{\tau\in H-H'} M_\tau) & \text{if} \quad \gamma=\gamma_0,\\ M_\gamma & \text{if} \quad \gamma\in H-H' \quad \text{and} \quad \gamma\neq\gamma_0. \end{cases}$$

It is clear, that

$$\bigcup_{\gamma<\omega_\alpha} L_\gamma$$

is a decomposition of the set S into mutually disjoint stationary sets.

Now we prove with the aid of Lemma 1 the following

Theorem 2. *If ω_α is an initial number with* $\mathrm{cf}(\alpha)>0$ *and the set of regular initial numbers $<\omega_{\mathrm{cf}(\alpha)}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$ then every stationary subset of $W(\omega_\alpha)$ may be expressed as the sum of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets.*

Proof. We distinguish two cases:

$$\text{a)}\quad \mathrm{cf}(\alpha)=\alpha, \qquad \text{b)}\quad \mathrm{cf}(\alpha)<\alpha.$$

Case a). Let S be an arbitrary stationary subset of $W(\omega_\alpha)$, and $\{\varrho_\nu\}_{\nu<\tau}$ $(\tau\leqq\omega_\alpha)$ the sequence of the regular initial numbers $<\omega_\alpha$ arranged according to their magnitude. Let us denote by P_ν the set of the limit numbers $<\omega_\alpha$ which are confinal to ϱ_ν. It is clear that the sets P_ν $(\nu<\tau)$ give decomposition of the set $W(\omega_\alpha)-\{\xi+1:\xi<\omega_\alpha\}$ into mutually disjoint sets. By our assumption the set $\{\varrho_\nu\}_{\nu<\tau}$ is non-stationary in $W(\omega_\alpha)$. Thus by Theorem I there exists an ordinal number $\nu_0 < \tau$ for which the set:

$$M = S \cap P_{\nu_0}$$

is stationary in $W(\omega_\alpha)$. By Lemma 1 the set M can be expressed as the sum of $\aleph_\alpha$ mutually disjoint stationary sets $M_\mu (\mu < \omega_\alpha)$. Put

$$N_\mu = \begin{cases} M_0 \cup (S - M) & \text{if} \quad \mu = 0, \\ M_\mu \quad \text{if} \quad \mu < \omega_\alpha & \text{and} \quad \mu \neq 0. \end{cases}$$

It is clear that

$$\bigcup_{\mu < \omega_\alpha} N_\mu$$

is a decomposition of S into $\aleph_\alpha$ mutually disjoint stationary sets.

Case b). Let Z be a closed subset of $W(\omega_\alpha)$ which is confinal to $W(\omega_\alpha)$ and the elements of which are greater than $\omega_{\mathrm{cf}(\alpha)}$. Further let $Z = \{z_\gamma\}_{\gamma < \omega_{\mathrm{cf}(\alpha)}}$ be a well-ordering of the elements of Z according to their magnitude such that $\lim_{\gamma < \tau} z_\gamma = z_\tau$ for every limit number $\tau < \omega_{\mathrm{cf}(\alpha)}$. Since the function $f(\xi) = z_\xi$ is increasing and continuous, $f(\xi)$ maps every subset of $W(\omega_{\mathrm{cf}(\alpha)})$ closed and confinal to $W(\omega_{\mathrm{cf}(\alpha)})$ into a subset of $W(\omega_\alpha)$ closed and confinal to $W(\omega_\alpha)$. It follows from this that the function $f(\xi)$ maps every stationary (or non-stationary) subset of $W(\omega_{\mathrm{cf}(\alpha)})$ into a stationary (or non-stationary) subset of $W(\omega_\alpha)$. Let $\{\varrho_\nu\}_{\nu < \tau}$ $(\tau \leqq \omega_{\mathrm{cf}(\alpha)})$ be the set of the regular initial numbers $< \omega_{\mathrm{cf}(\alpha)}$ arranged according to their magnitude. Let us denote by Γ_ν the set of the limit numbers $< \omega_{\mathrm{cf}(\alpha)}$ which are confinal to ϱ_ν. Put

$$Q_0 = \{z_{\xi+1} : \xi < \omega_{\mathrm{cf}(\alpha)}\}, \qquad P_\nu = \{z_\xi : \xi \in \Gamma_\nu\}.$$

It is clear that

$$\bigcup_{\nu < \tau} P_\nu \qquad (\tau \leqq \omega_{\mathrm{cf}(\alpha)})$$

is a decomposition of the set $Z - Q_0$ into mutually disjoint sets P_ν $(\nu < \tau)$. By our assumption the set $\{\varrho_\nu\}_{\nu < \tau}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$. Thus the set $\{z_{\varrho_\nu}\}_{\nu < \tau}$ is non-stationary in $W(\omega_\alpha)$. It is easy to see that the first element of the set P_ν is z_{ϱ_ν}. By Theorem I there exists an ordinal number $\nu_0 < \tau$ for which the set

$$M = S \cap P_{\nu_0}$$

is stationary in $W(\omega_\alpha)$. According to Lemma 1 the set Γ_ν can be expressed as the sum of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint in $W(\omega_{\mathrm{cf}(\alpha)})$ stationary sets. Consequently the set M can be expressed as $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint sets M_μ $(\mu < \omega_{\mathrm{cf}(\alpha)})$, stationary in $W(\omega_\alpha)$. Put

$$N_\mu = \begin{cases} M_0 \cup (S - M) & \text{if} \quad \mu = 0, \\ M_\mu \quad \text{if} \quad \mu < \omega_{\mathrm{cf}(\alpha)} & \text{and} \quad \mu \neq 0. \end{cases}$$

It is clear that

$$\bigcup_{\mu < \omega_{\mathrm{cf}(\alpha)}} N_\mu$$

is a decomposition of S into $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint sets, stationary in $W(\omega_\alpha)$.

Corollary 3. *If ω_α is an initial number with* $\mathrm{cf}(\alpha) = \gamma + 1$, *then every stationary subset of $W(\omega_\alpha)$ can be expressed as $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets.*

Proof. The set R of the regular initial numbers smaller than $\omega_{\gamma+1}$ has power $\leqq \aleph_\gamma$, consequently the set R is non-stationary in $W(\omega_{\gamma+1})$.

It follows from Theorem 2 with the aid of Theorem I the following

Theorem 4. *If S is a stationary subset of $W(\omega_\alpha)$, $cf(\alpha)>0$, and the set of the regular initial numbers $<\omega_{cf(\alpha)}$ is non-stationary in $W(\omega_{cf(\alpha)})$, then the set S can be expressed as the sum $\bigcup_{\eta<\omega_{cf(\alpha)}} S_\eta$ of $\aleph_{cf(\alpha)}$ mutually disjoint stationary sets S_η such that for each stationary subset M of S there is an ordinal number $\eta_0<\omega_{cf(\alpha)}$ for which $M \cap S_{\eta_0}$ is a stationary set.*

II.

Suppose we are given, for each countable limit ordinal number ξ, a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\dots$ converging to ξ. B. Rotman [4] has proved that, for all but finitely many positive integers i, each function f_i takes $\aleph_1$ different values, $\aleph_1$ times each; i.e. there is a set S_i of power $\aleph_1$ such that the set $\{\xi : f_i(\xi)=\gamma\}$ has power $\aleph_1$ for each $\gamma \in S_i$.

We prove in this paper the following more general result.

Theorem 5. *If for every limit number $\xi \in W(\omega_1)$ there exists a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\dots<f_i(\xi)<\dots$ converging to ξ then for all but finitely many positive integers i there is a set S_i of power $\aleph_1$ such that the set $\{\xi : f_i(\xi)=\gamma\}$ is stationary in $W(\omega_1)$ for each $\gamma \in S_i$.*

First we prove the following.

Lemma 6. *Let ω_α be a regular initial number with $\alpha>0$, S a stationary subset of $W(\omega_\alpha)$, and $f(\xi)$ a regressive function on S, then the difference $S-U$, where U is the union of those sets $\{\xi : f(\xi)=\gamma\}$ $(\gamma \in f(S))$ which are stationary in $W(\omega_\alpha)$, is non stationary in $W(\omega_\alpha)$.*

Proof. Let $\{\eta_\nu\}_{\nu<\tau}$ $(\tau \leqq \omega_\alpha)$ be the set of the ordinal numbers $\eta \in f(S)$ arranged according to their magnitude, for which the sets $K_\eta=\{\xi \in S : f(\xi)=\eta\}$ are non-stationary in $W(\omega_\alpha)$. Let us denote by ξ_ν the first element of the set K_{η_ν}. By Theorem III the set $\{\xi_\nu\}_{\nu<\tau}$ is non-stationary in $W(\omega_\alpha)$ since for the function

$$g(\xi_\nu)=\eta_\nu$$

the relation $g(\xi_\nu)\neq g(\xi_\tau)$ holds if $\nu \neq \tau$. Thus by Theorem I the set

$$K=\bigcup_{\nu<\tau} K_{\eta_\nu}$$

is non-stationary in $W(\omega_\alpha)$. Since $S-K$ is equal to the union of those sets $\{\xi : f(\xi)=\gamma\}$ $(\gamma \in f(S))$ which are stationary in $W(\omega_\alpha)$, the lemma is proved.

Proof of Theorem 5. It is clear that, for every $i<\omega$, the domain of the function f_i is the set L of the limit numbers $\xi \in W(\omega_1)$. Let us denote by L_i the union of the sets of the form $\{\xi : f_i(\xi)=\gamma\}$ $(\gamma \in f_i(S))$ which are stationary in $W(\omega_1)$. By Lemma 6, there exists a non-stationary set H_i for every $i<\omega$ such that $L_i = L-H_i$. Since $\omega<\omega_1$, the set

$$H=\bigcup_{i<\omega} H_i$$

is, by Theorem I, non-stationary in $W(\omega_1)$. Thus the set $L-H$ contains a closed subset Z which is confinal to $W(\omega_1)$. Since

$$L-\bigcup_{i<\omega} H_i = \bigcap_{i<\omega}(L-H_i),$$

the relation

$$Z\subset \bigcap_{i<\omega}(L-H_i) = \bigcap_{i<\omega} L_i$$

holds. Let $\{\lambda_\nu^{(i)}\}_{\nu<\tau^{(i)}}$ $(i<\omega)$ be the set of the elements $\lambda\in f_i(L)$ for which $K_\gamma = \{\xi\in L: f_i(\xi)=\lambda\}$ is stationary in $W(\omega_1)$. It is clear that

$$L_i = \bigcup_{\nu<\tau^{(i)}} K_{\lambda_\nu^{(i)}}.$$

We show now that the assumption $\tau^{(i)}<\omega_1$ for all i leads to a contradiction. If the inequality $\tau^{(i)}<\omega_1$ holds for every $i<\omega$ then the power of the set

$$\Gamma = \bigcup_{i<\omega}\{\lambda_\nu^{(i)}\}_{\nu<\tau^{(i)}}$$

is smaller than $\aleph_1$. In this case there is an ordinal number $\gamma<\omega_1$ which is greater than each element of the set Γ. Now if

$$\mu\in Z\subset \bigcap_{i<\omega} L_i,$$

then the relation

$$f_i(\mu)\in\Gamma$$

holds for each $i<\omega$; consequently

$$\mu=\lim f_i(\mu)\leqq\gamma.$$

It follows from this that each element $\mu\in Z$ is smaller than $\gamma+1$ which is impossible since the set Z is confinal to $W(\omega_1)$.

We show now that the relation $\tau^{(i)}=\omega_1$ holds for all but finitely many integers i. In the contrary case there would exist an increasing sequence $\{i_j\}_{j<\omega}$ of natural numbers such that $\tau^{(i_j)}<\omega_1$ for each $j<\omega$. But then we could apply to this sequence the preceding arguments thus obtaining a j for which $\tau^{(i_j)}=\omega_1$. This contradiction finishes the proof of Theorem 5.

References

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