On a process concerning inaccessible cardinals. II

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This paper is a continuation of reference I (see [1]), in which a process concerning inaccessible cardinals has been defined. In this paper we freely make use of the notations, definitions, and theorems of [1].

From now on, in the definition of the process, we start with strongly inaccessible initial numbers. This means that the values of the function $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\eta)})$ are strongly inaccessible numbers.

First we prove the following

Theorem 2. *If $\alpha = n_{\eta,\eta}(0)$ and $\eta < \alpha$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\eta)}) < \alpha$ is non-stationary in α.*

Proof. We may assume by Theorem 1 of [1] that $\eta \geqq \omega$. Denote by $\gamma(\beta)$ the value $f_\eta(0, \dots, 0, \dots, \beta)$. As the first step we prove the following statement.

(j_1) *Suppose that $\beta \neq 0$. Then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \gamma(\beta), \psi^{(\mu+1)}, \dots, \psi^{(\xi)}, \dots, \psi^{(\eta)})$$

for every $\mu < \eta$, provided that $\psi^{(\eta)} < \beta$ and $\psi^{(\xi)} < \gamma(\beta)$ for each ξ $(\mu + 1 \leqq \xi < \eta)$.

To prove this statement, we write η in the form $\eta = \omega\xi + n$, where $\xi \leqq \eta$ and $0 \leqq n < \omega$.

We distinguish the cases $n = 0$ and $n > 0$.

Case $n = 0$. We prove the following three statements, the third of which immediately implies (j_1):

(a) *If $\nu < \beta$ and $\tau < \eta$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \overset{(\tau)}{\gamma(\beta)}, 0, \dots, 0, \dots, \nu).$$

(b) *If $\nu < \beta$, $\sigma < \xi$ and $0 < m < \omega$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \dots, \psi^{(\omega\sigma+l)}, \dots, \psi^{(\omega\sigma+m)}, 0, \dots, 0, \dots, \nu),$$

provided that $\psi^{(\omega\sigma+l)} < \gamma(\beta)$ for each l $(1 \leqq l \leqq m)$.

(c) *If $\nu < \beta$, $0 < \sigma < \xi$, $\varkappa < \omega\sigma$ and $0 \leqq m < \omega$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \overset{(\varkappa)}{\gamma(\beta)}, 0, \dots, 0, \dots, \psi^{(\omega\sigma)}, \dots, \psi^{(\omega\sigma+l)}, \dots, \psi^{(\omega\sigma+m)}, 0, \dots, 0, \dots, \nu),$$

provided that $\psi^{(\omega\sigma+l)} < \gamma(\beta)$ for each l $(0 \leqq l \leqq m)$.

Ad (a): Since $\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \beta)$, we have

$$(19) \qquad \gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \beta).$$

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$(20) \qquad Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varkappa) = \bigcap_{\nu<\varkappa} Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu),$$

where $\varkappa$ is a limit number,

$$(21) \qquad Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varrho+1) = \bigcap_{\tau<\eta} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \varrho),$$

and

$$(22) \qquad f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau+1)}{1}, 0, \ldots, 0, \ldots, \nu) = \left(f_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu)\right)'.$$

With the help of (19), (20) and (21) we obtain

$$(23) \qquad \gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \nu)$$

for every $\nu\leqq\beta$; moreover, (23) and (21) imply

$$(24) \qquad \gamma(\beta)\in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$ and for every $\tau<\eta$. From this we conclude that (a) is valid. For if not, then there are three ordinal numbers $\nu_0<\beta$, $\tau_0<\eta$ and $\varrho_0<\gamma(\beta)$ such that

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\tau_0)}{\varrho_0}, 0, \ldots, 0, \ldots, \nu_0).$$

Hence, by (22), we have

$$\gamma(\beta)\notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau_0+1)}{1}, 0, \ldots, 0, \ldots, \nu_0).$$

Thus, by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$, we obtain

$$\gamma(\beta)\notin Rf_\eta(0, \ldots, 0, \ldots, 0, \alpha^{(\tau_0+1)}, 0, \ldots, 0, \ldots, \nu_0),$$

which contradicts the fact that (24) is valid for every $\nu<\beta$ and $\tau<\eta$.

Ad (b): From (a) we get

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$. Hence

$$(25) \qquad \gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$.

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$(26) \qquad \begin{aligned} &Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu) = \\ &= \bigcap_{\mu<\gamma(\beta)} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu) \end{aligned}$$

and

(27)
$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\mu+1}, 0, \ldots, 0, \ldots, \nu) =$$
$$=(f_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu))'.$$

By (25) and (26) we have

(28)
$$\gamma(\beta) \in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu)$$

for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. First we show that $\gamma(\beta)$ satisfies the equality

(29)
$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\omega\sigma+m-1)}{\gamma(\beta)}, \mu, 0, \ldots, 0, \ldots, \nu)$$

for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. If not, then there are two ordinal numbers $\mu_0<\gamma(\beta)$ and $\varrho_0<\gamma(\beta)$ such that

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \varrho_0, \overset{(\omega\sigma+m)}{\mu_0}, 0, \ldots, 0, \ldots, \nu).$$

Hence, by (27)

$$\gamma(\beta) \notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\mu_0+1}, 0, \ldots, 0, \ldots, \nu).$$

On the other hand it follows from this and the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$\gamma(\beta) \notin Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu_0+1, 0, \ldots, 0, \ldots, \nu),$$

which contradicts the fact that (28) holds for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. Thus we conclude that $\gamma(\beta)$ satisfies (29) for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$.

Let now l be a natural number for which $0<l<m$. Assume that whenever $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(i=l+1, \ldots, m)$ then

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since $\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \overset{(m\sigma+m-1)}{\gamma(\beta)}, \mu, 0, \ldots, 0, \ldots, \nu)$ for every $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and for every $\mu<\gamma(\beta)$ it remains to prove that this assumption implies that whenever $\nu<\beta$ $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l\leqq i\leqq m)$ then

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that, for given σ, ν, $0<m<\omega$, $\psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}$ the equalities

(30)
$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$
$$= \bigcap_{\mu<\gamma(\beta)} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu)$$

and

$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$

(31)

$$= (f_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu)$$

hold.

By (30) and (31) we obtain for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l+1\leqq i\leqq m)$ that

(32)

$$\gamma(\beta)\in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Now we show for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $0<\eta$, $0<m<\omega$, $\psi^{(\omega\sigma+i)}<$ $<\gamma(\beta)$ $(l+1\leqq i\leqq m)$ that the ordinal number $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

In the contrary case there are two ordinal numbers $\mu_0<\gamma(\beta)$ and $\tau_0<\gamma(\beta)$ such that

$$\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \tau_0, \mu_0, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Hence, by (30), we have

$$\gamma(\beta)\notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu_0+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Consequently, by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$

$$\gamma(\beta)\notin Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu_0+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since $\gamma(\beta)$ is a limit number, we have $\mu_0+1<\gamma(\beta)$, which contradicts the fact that (32) holds for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l+1\leqq i\leqq m)$. Thus we may conclude that the statement (b) is true.

Ad (c): If $\nu<\beta$, $\sigma<\xi$ and $0<m<\omega$ then, by (b), $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

provided that $\psi^{(\omega\sigma+l)}<\gamma(\beta)$ for each l $(1\leqq l\leqq m)$. It follows from this, under the same conditions, that

$$\gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since, by the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$

$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$
$$= \bigcap_{\mu<\gamma(\beta)} Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu+1, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$
$$= \bigcap_{\tau<\omega\sigma} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \mu, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

and

$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau+1)}{1}, 0, \ldots, 0, \ldots, \nu) = (f_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu))',$$

we can apply the method used in the proof of (a). Thus we obtain the proof of (c).

Case $n>0$. By the same argument as in the proof of (a) and (b) we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\gamma(\beta),\psi^{(\omega\xi+1)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\omega\xi+n)})$$

for any $\psi^{(\omega\sigma+l)}<\gamma(\beta)$ $(1\leqq l\leqq n-1)$ and $\psi^{(\eta)}<\beta$.

Hence, by the argument used in the proof of (c), we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\overset{(\varkappa)}{\gamma(\beta)},0,\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\omega\xi+n)})$$

for any $\varkappa<\omega\xi$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$ and $\psi^{(\eta)}<\beta$.

From this, by the argument applied in the proof of (b), we conclude that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\gamma(\beta),\psi^{(\omega\sigma+1)},\ldots,\psi^{(\omega\sigma+k)},\ldots,\psi^{(\omega\sigma+m)},0,\ldots$$
$$\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\eta)})$$

whenever $0<m<\omega$, $\sigma<\xi$, $\psi^{(\omega\sigma+k)}<\gamma(\beta)$ $(0<k\leqq m)$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$, and $\psi^{(\eta)}<\beta$.

Finally, by the argument of the proof of (c), we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\overset{(\varkappa)}{\gamma(\beta)},0,\ldots,0,\ldots,\psi^{(\omega\sigma)},\ldots,\psi^{(\omega\sigma+k)},\ldots$$
$$\ldots,\psi^{(\omega\sigma+m)},0,\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\sigma+l)},\ldots,\psi^{(\eta)})$$

whenever $m<\omega$, $\varkappa<\omega\sigma$, $\psi^{(\omega\sigma+k)}<\gamma(\beta)$ $(0\leqq k\leqq m)$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$, and $\psi^{(\eta)}<\beta$. This immediately implies the statement (j_1) in the case $n>0$ too.

The same method can be used to prove the following statement:

(j_2) *Assume that* $\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)}$ $(0<\mu\leqq\eta)$ *are given ordinal numbers and* $\underline{\alpha}^{(\eta)}\neq0$. *Then* $\gamma=f_\eta$ $(0,\ldots,0,\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)})$ *satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\mu)},\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

or every τ $(0\leqq\tau\leqq\mu)$ *provided that* $\psi^{(\mu)}<\underline{\alpha}^{(\mu)}$ *and* $\psi^{(\xi)}<\gamma$ *for each* ξ $(\tau+1\leqq\xi<\mu)$.

Now we proceed to prove the following statement:

(j_3) *Assume that* $\underline{\alpha}^{(0)},\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)}$ $(0\leqq\mu\leqq\eta)$ *are given ordinal numbers,* $\underline{\alpha}^{(0)}\neq0$ *and* $\underline{\alpha}^{(\mu)}\neq0$. *Then* $\gamma=f_\eta(\underline{\alpha}^{(0)},0,\ldots,0,\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)})$ *satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\mu)},\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every τ $(0\leqq\tau<\mu)$, *provided that* $\psi^{(\mu)}<\underline{\alpha}^{(\mu)}$ *and* $\psi^{(\xi)}<\gamma$ *for each* ξ $(\tau+1\leqq\xi<\mu)$.

Let us denote λ the ordinal number $\underline{\alpha}^{(\mu)}$. Consider first the case when μ is an ordinal number of the first kind. It follows from the definition of $f_\eta(\alpha^{(0)},\alpha^{(1)},\ldots,\alpha^{(\eta)})$ that

$$f(\alpha^{(0)},0,\ldots,0,\ldots,\varrho+1,\underline{\alpha}^{(\mu+1)}\ldots,\underline{\alpha}^{(\eta)})=(f_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\varrho,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for $\lambda=\varrho+1$ and

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\lambda,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=\bigcap_{\nu<\lambda}Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for a limit number λ. These imply that for every $\nu<\lambda$

$$\gamma\in Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

Hence we easily conclude that

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

Thus, by (j_2), we get (j_3) in the case where μ in an ordinal number of the first kind.

Suppose now that μ is a limit number. Then from the definition of $f_\eta(\alpha^{(0)},\alpha^{(1)},\ldots,\alpha^{(\eta)})$ we see that

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\varrho+1,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=$$
$$=\bigcap_{\xi<\mu}Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\xi)},0,\ldots,0,\ldots,\varrho,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for $\lambda=\varrho+1$ and

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\lambda,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=$$
$$=\bigcap_{\nu<\lambda}Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for a limit number λ. By a proof analogous to that of (b) and (c), we obtain (j_3) in the case where μ is a limit number.

Now we can prove the following statement:

(j_4) *Let* $\{\varkappa_\zeta\}_{\zeta\leqq\sigma}$ $(\sigma\leqq\eta)$ *be the strictly increasing sequence of the ordinal numbers* $\varkappa\leqq\eta$ *for which* $\alpha^{(\varkappa)}\neq 0$. *Assume that* $\varkappa_0=0$. *Then* $\gamma=f_\eta(\underline{\alpha}^{(0)},\underline{\alpha}^{(1)},\ldots,\underline{\alpha}^{(\eta)})$ *satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every ζ $(1\leqq\zeta\leqq\sigma)$ *and for every* τ $(0\leqq\tau\leqq\varkappa_\zeta)$, *provided that* $\psi^{(\varkappa_\zeta)}<\underline{\alpha}^{(\varkappa_\zeta)}$ *and* $\psi^{(\xi)}<\gamma$ *for each* ξ $(\tau+1\leqq\xi<\varkappa_\zeta)$.

Indeed, if (j_4) is true for a fixed ζ $(0<\zeta\leqq\sigma)$, then

$$\gamma=f_\eta(\gamma,0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

If we apply (j_3) to $\underline{\alpha}^{(0)}=\gamma$, we obtain that

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every τ $(0\leqq\tau\leqq\varkappa_\zeta)$, provided that $\psi^{(\varkappa_\zeta)}<\underline{\alpha}^{(\varkappa_\zeta)}$ and $\psi^{(\xi)}<\gamma$ for each ξ $(\tau+1\leqq\leqq\xi<\varkappa_\zeta)$. This proves the statement (j_4).

Now we proceed the proof of Theorem 2 by showing that the set

$$Rf_\eta(0,\ldots,0,\ldots,\beta)/\alpha \tag{3}$$

is non-stationary in α. We define a function g on $M=Rf_\eta(0,\ldots,0,\ldots,\beta)/\alpha$ by writing

$$g\big(f_\eta(0,\ldots,0,\ldots,\beta)\big)=\beta.$$

Since $f_\eta(0, \ldots, 0, \ldots, \tau)$ is a strictly increasing function of the variable τ and for every $\beta < \alpha$ the inequality

$$\beta < f_\eta(0, \ldots, 0, \ldots, \beta)$$

holds, we obtain that the function g is strictly divergent and regressive on M. Therefore Theorem I (see [1]) implies that the set (33) is non-stationary in α.

Next we prove, by transfinite induction, the following statement.

(j_5) *For every* μ, $0 < \mu \leqq \eta$ *the set*

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α, *where* $\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}$ *are given ordinal numbers* $< \alpha$.

First we show that the set

$$N = Rf_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. We define a function g on N by writing

$$g\big(f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})\big) = \alpha_\xi^{(0)}.$$

From the definition of $\alpha_\xi^{(0)}(\underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$ and $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$, we obtain

$$\alpha_\xi^{(0)} < f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$$

and

$$f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(\alpha_{\xi+1}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

From these we infer that the function g is strictly divergent and regressive on N and, therefore, by Theorem I ([1]), we obtain that the set N is non-stationary in α.

Let ν be a given ordinal number and suppose that for every μ $(1 \leqq \mu < \nu)$ the set

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α.

There are two cases:

a) ν is an ordinal number of the first kind, i.e. $\nu = \tau + 1$,

b) ν is an ordinal number of the second kind.

Case a): We show that the set

$$L = Rf_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. We define a function g on L by writing

$$g\big(f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})\big) = \alpha_\varphi^{(\tau)}.$$

From the definition of $\alpha^{(\tau)}(\underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$ and $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ we obtain

$$\alpha_\varphi^{(\tau)} < f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$$

and

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \alpha_{\varphi+1}^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

From these we conclude that the function g is strictly divergent and regressive on

L, and, therefore, by Theorem I ([1]), we obtain that the set L is non-stationary in α. It follows from the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}) \leqq f_\eta(\alpha_\xi^{(0)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

By our assumption, for given $\underline{\alpha}_\varphi^{(\tau)}$ the set

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \underline{\alpha}_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. On the other hand it is easy to verify that for any two different elements $\underline{\alpha}_\varphi^{(\tau)}$ and $\underline{\alpha}_\sigma^{(\tau)}$ the sets

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

and

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}_\sigma^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

have no common elements. Since the set of the first elements of the sets

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

with $\alpha_\varphi^{(\tau)} \in A_{\tau,\eta}(\underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$ is equal to L we obtain from Theorem II ([1]) that the union of these sets is non-stationary in α.

Case b): Put

$$Q_{\mu,\nu,\eta} = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \underline{\alpha}_\varkappa^{(\mu)}, \ldots, \underline{\alpha}_\varrho^{(\delta)}, \ldots, \underline{\alpha}^{(\nu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha,$$

where $\underline{\alpha}_\varrho^{(\delta)}$ is fixed for each δ $(\mu \leqq \delta < \nu)$. It is easy to see that

$$Q_{1,\nu,\eta} \subset Q_{2,\nu,\eta} \subset \ldots \subset Q_{\mu,\nu,\eta} \subset \ldots \quad (\mu < \nu).$$

By the hypothesis the set $Q_{\mu,\nu,\eta}$ $(\mu < \nu)$ is non-stationary in α. Since $\mu < \nu \leqq \eta < \alpha$ by Theorem III ([1]), we obtain that the set

$$\bigcup_{\mu<\nu} Q_{\mu,\nu,\eta} = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\mu)}, \ldots, \underline{\alpha}^{(\nu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. Thus the statement (j_5) is proved.

Since the set $Rf_\eta\,(0, \ldots, 0, \ldots, \beta)/\alpha$ is non-stationary in α, we obtain from (j_5) that the set

$$K = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$$

is non-stationary in α.

Consider now an arbitrary element $\gamma = f_\eta(\underline{\alpha}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$ of K. Let $\{\varkappa_\zeta\}_{\zeta \leqq \sigma}$ $(\sigma \leqq \eta)$ be the strictly increasing sequence of the ordinal numbers $\varkappa$, $0 \leqq \varkappa \leqq \eta$, for which $\alpha^{(\varkappa)} \neq 0$. Let us denote by ζ_0 the smallest ordinal number $\zeta \leqq \sigma$ for which $\varkappa_\zeta \geqq 2$. Then the statements (j_1)—(j_5) imply that

$$\gamma = f_\eta(0, \ldots, 0, \ldots, \gamma, \psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)}, \underline{\alpha}^{(\varkappa_\zeta+1)}, \ldots, \underline{\alpha}^{(\eta)}) \tag{34}$$

for every ζ $(\zeta_0 \leqq \zeta \leqq \sigma)$ and τ $(0 \leqq \tau \leqq \varkappa_\zeta)$, provided that $\psi^{(\varkappa_\zeta)} < \underline{\alpha}^{(\varkappa_\zeta)}$ and $\psi^{(\xi)} < \gamma$ for each ξ $(\tau + 1 \leqq \xi < \varkappa_\zeta)$.

Let us denote by $S_{\zeta,\tau}$, where $\zeta_0 \leqq \zeta \leqq \sigma$ and $0 \leqq \tau \leqq \varkappa_\zeta$, the set of the sequences

$$(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$$

such that $\psi^{(\varkappa_\zeta)} < \alpha^{(\varkappa_\zeta)}$ and $\psi^{(\xi)} < \gamma$ for each ξ $(\tau + 1 \leqq \xi < \varkappa_\zeta)$. Since $\eta < \alpha$ and α is a strongly inaccessible initial number, the power of the set $S_{\zeta,\tau}$ is smaller than α.

It follows from the statement (j_5) that for any element $(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$ of $S_{\zeta,\tau}$ the set

$$C(\psi^{(\tau+1)}, \ldots, \psi^{(\varkappa_\zeta)}) =$$
$$= Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\delta)}, \ldots, \gamma, \psi^{(\tau+1)}, \ldots, \psi^{(\varkappa_\zeta)}, \underline{\alpha}^{(\varkappa_\zeta+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α.

Since $\zeta \leqq \sigma \leqq \eta < \alpha$ and α is a strongly inaccessible initial number and hence the power of $S_{\zeta,\tau}$ is smaller than α, Theorem III ([1]) implies that the set

$$B(\gamma) = \bigcup_{\zeta \leqq \sigma} \bigcup_{\tau < \varkappa_\zeta} \bigcup_{\psi^{(\tau+1)} < \gamma} \cdots \bigcup_{\psi^{(\xi)} < \gamma} \cdots \bigcup_{\psi^{(\varkappa_\zeta)} < \underline{\alpha}^{(\varkappa_\zeta)}} C(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$$

is non-stationary in α. On the other hand, by (34), the smallest element of the set $B(\gamma)$ is γ.

In this manner, with every element $\gamma = f_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\eta)})$ of K we have associated a non-stationary set $B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\eta)})$ the smallest element of which is γ.

It only remains to prove that

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. Since K is non-stationary in α, the sets

$$B_0 = Rf_\eta(\alpha_\xi^{(0)}, \underline{\alpha}_\zeta^{(1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$B_1 = Rf_\eta(0, \alpha_\zeta^{(1)}, \underline{\alpha}_\psi^{(2)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$\vdots$$
$$B_\mu = Rf_\eta(0, \ldots, 0, \ldots, \alpha_\varrho^{(\mu)}, \underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$\vdots$$

are non-stationary in α, where $\underline{\alpha}_\zeta^{(1)}, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)}$ are fixed ordinal numbers $< \alpha$.

Let $\nu > 1$ be a given ordinal number, and suppose that for every μ $(1 \leqq \mu < \nu)$ the set

$$D(\underline{\alpha}_\varrho^{(\mu)}, \underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)}) = \tag{35}$$
$$= \bigcup_{\substack{\alpha_\xi^{(0)} \in A_{0,\eta} \\ \alpha_\xi^{(0)} < \alpha}} \cdots \bigcup_{\substack{\alpha_\psi^{(\vartheta)} \in A_{\vartheta,\eta} \\ \alpha_\psi^{(\vartheta)} < \alpha}} \cdots B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\vartheta)}, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$$

is non-stationary in α. We must prove that the set

$$D(\underline{\alpha}_\varphi^{(\nu)}, \ldots, \underline{\alpha}_\delta^{(\eta)}) = \tag{36}$$
$$= \bigcup_{\substack{\alpha_\xi^{(0)} \in A_{0,\eta} \\ \alpha_\xi^{(0)} < \alpha}} \cdots \bigcup_{\substack{\alpha_\psi^{(\vartheta)} \in A_{\vartheta,\eta} \\ \alpha_\psi^{(\vartheta)} < \alpha}} \cdots B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\vartheta)}, \ldots, \underline{\alpha}_\varphi^{(\nu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$$

is non-stationary in α. It is easy to verify that the smallest element of (35) is $f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$. Thus the set of the first elements of the sets

$D(\alpha_{\varrho}^{(\mu)}, \underline{\alpha}_{\varphi}^{(\mu+1)}, \ldots, \underline{\alpha}_{\delta}^{(\eta)})$ with $\alpha_{\varrho}^{(\mu)} \in A_{\mu,\eta}(\underline{\alpha}_{\varphi}^{(\mu+1)}, \ldots, \underline{\alpha}_{\delta}^{(\eta)})$ is equal to B_μ. Suppose now that ν is a number of the first kind, i.e. $\nu = \vartheta + 1$. In this case Theorem IV ([1]) implies that the set

$$\bigcup_{\alpha_{\sigma}^{(\vartheta)} \in A_{\vartheta,\eta},\ \alpha_{\sigma}^{(\vartheta)} < \alpha} D(\alpha_{\sigma}^{(\vartheta)}, \underline{\alpha}_{\varphi}^{(\vartheta+1)}, \ldots, \underline{\alpha}_{\delta}^{(\eta)})$$

is non-stationary in α. Suppose now that ν is a limit number. For the proof of our statement it is sufficient to show that the set

$$\bigcup_{\mu<\nu} D(\underline{\alpha}_{\varrho}^{(\mu)}, \ldots, \underline{\alpha}_{\delta}^{(\eta)}) \qquad (\nu \leqq \eta < \alpha)$$

is non-stationary in α. But this follows from the hypothesis and from Theorem IV. Thus the proof of Theorem 2 is complete.

In an entirely analogous way it may be proved the following

Theorem 3. *If $\eta < \alpha$, $\mu < \eta$ and $\alpha = n_{\mu,\eta}(\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots \underline{\alpha}^{(\eta)}) < \alpha$ is non-stationary in α.*

We prove now the following

Theorem 4. *If $\alpha = f_\eta(\underline{\alpha}^{(0)}, \ldots, \underline{\alpha}^{(\xi)} \ldots, \underline{\alpha}^{(\eta)})$, $\eta < \alpha$, and $\underline{\alpha}^{(\xi)} < \alpha$ for each $\xi \leqq \eta$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}\ \alpha^{(1)}, \ldots, \alpha^{(\eta)}) < \alpha$ is non-stationary in α.*

Proof. Let $\{\xi_\zeta\}_{\zeta \leqq \sigma}$ $(\sigma \leqq \eta)$ be the strictly increasing sequence of the ordinal numbers $\xi \leqq \eta$ for which $\underline{\alpha}^{(\xi)} \neq 0$.

Put

$$\gamma(\nu^{(\xi_\mu)}) = f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_\mu)}, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)}),$$

where $\nu^{(\xi_\mu)} < \underline{\alpha}^{(\xi_\mu)}$ if $\mu = 0$ and $\nu^{(\xi_\mu)} \leqq \underline{\alpha}^{(\xi_\mu)}$ if $0 < \mu \leqq \sigma$.

First we show that the set

$$\{f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_0)}, \ldots, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}\}_{\nu(\xi_0) < \alpha(\xi_0)} \tag{37}$$

is non-stationary in α. Indeed, if $\underline{\alpha}^{(\xi_0)} = \underline{\nu}^{(\xi_0)} + 1$ then

$$f_\eta(0, \ldots, 0, \ldots, \underline{\nu}^{(\xi_0)}, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, \ldots, \underline{\alpha}^{(\eta)});$$

moreover, if $\underline{\alpha}^{(\xi_0)}$ is a limit number, then

$$\lim_{\nu^{(\xi_0)} < \underline{\alpha}^{(\xi_0)}} f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_0)}, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}) < \alpha,$$

because $\underline{\alpha}^{(\xi_0)} < \alpha$ and α are regular. This implies that the set (37) is non-stationary in α.

Now we show that for every μ $(0 < \mu \leqq \sigma)$ the set

$$\{f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_\mu)}, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)})\}_{\nu(\xi_\mu) \leqq \underline{\alpha}(\xi_\mu)} \tag{38}$$

is non-stationary in α. Indeed, if $\mu > 0$ then

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}),$$

on the other hand

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, 0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}) \leqq f_\eta(\underline{\alpha}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}) = \alpha.$$

Hence, for $\mu > 0$,

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}) < \alpha.$$

Consequently, the set (38), where $0 < \mu \leqq \sigma$ is non-stationary in α.

We may suppose without loss of generality that $\xi_0 = 0$. [In virtue of (j_5) and the non-stationarity of the sets (38) with $0 < \mu \leqq \sigma$, the set

$$\bigcup_{0<\mu\leqq\sigma} \bigcup_{\nu(\xi_\mu)\leqq\underline{\alpha}(\xi_\mu)} Rf_\eta(\alpha_\xi^{(0)}, \ldots, \alpha_\varphi^{(\delta)}, \ldots, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha \tag{39}$$

is non-stationary in α. Applying to the set (39) the argument used for the set $Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$ after the proof of (j_5) in the proof of Theorem 2, we obtain Theorem 4.

Remark. If in the definition of the process we start with weakly inaccessible initial numbers then we can only prove Theorems 2 (see [1]), 3, and 4 for $\eta < \omega$.

We prove now the following

Theorem 5. *If α is the smallest ordinal number of η for which $\eta = f_\eta(0, \ldots, 0, \ldots, 1)$ then the set of the ordinal numbers of the form $f_\tau(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\tau)})$, where $\tau < \eta$, is non-stationary in α.*

Proof. First we show that the set $N = \{f_\varrho(0, \ldots, 0, \ldots, 1)\}_{\varrho<\alpha}$ is non-stationary in α.

Since α is the smallest ordinal number of η for which $\eta = f_\eta(0, \ldots, 0, \ldots, 1)$, the relation

$$\varrho < f_\varrho(0, \ldots, 0, \ldots, 1) \tag{40}$$

holds for each $\varrho < \alpha$. By the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ we have

$$f_\varrho(0, \ldots, 0, \ldots, 1) < f_{\varrho+1}(0, \ldots, 0, \ldots, 1). \tag{41}$$

Let us define the function g on the set N by writing

$$g(f_\varrho(0, \ldots, 0, \ldots, 1)) = \varrho.$$

It follows from (40) and (41) that the function g is strictly divergent and regressive on the set N. Therefore, by Theorem I ([1]), the set N is non-stationary in α.

Consider the set

$$Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\varrho < \alpha). \tag{42}$$

Since, as by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ the equalities

$$\begin{aligned} f_\alpha(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0) &= f_0(\alpha^{(0)}), \\ f_\alpha(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, 0) &= f_1(\alpha^{(0)}, \alpha^{(1)}), \\ &\vdots \\ f_\alpha(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\varrho)}, 0, \ldots, 0, \ldots, 0) &= f_\varrho(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\varrho)}), \\ &\vdots \end{aligned}$$

hold, we may assume $\alpha_\delta^{(\varrho)} \geqq 1$ in (42).

With the help of (j_5) we get for given ϱ that the set

$$M_\varrho = Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\varrho < \alpha, \quad \alpha_\delta^{(\varrho)} \geqq 1)$$

is non-stationary in α. But the set $\{f_\varrho(0, \ldots, 0, \ldots, 1)\}_{\varrho<\alpha}$ of the first elements of the sets M_ϱ with $\varrho < \alpha$ is non-stationary in α. Therefore, making use of (j_5), the set

$$\bigcup_{\varrho<\alpha} Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\alpha_\delta^{(\varrho)} \geqq 1) \tag{43}$$

is non-stationary in α. Applying the same argument to the set (43) as in the proof of Theorem 2, after the proof of (j_5) for the set $Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$, Theorem 5 will be proved.

Reference

[1] G. Fodor, On a process concerning inaccessible cardinals. I, *Acta Sci. Math.*, **27** (1966), 111—124.

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