A type of extension of Banach spaces

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The purpose of this article is the study of a certain relation between a Banach space X and a linear subspace Y. A subset F of the unit ball of X^* is called a *Y-boundary* if for each y in Y, $\|y\| = \sup\{|f(y)| : f \in F\}$; X is a *bound extension* of Y if every Y-boundary is an X-boundary. By requiring that X itself admit no bound extension one has an obverse to the "projective resolution" of a compact Hausdorff space defined and constructed by GLEASON [1; Theorem 3. 2]. We believe, moreover, that the theory of P_1 spaces is naturally included in the present discussion.

Banach spaces may be real or complex; but since certain arguments are more difficult in the complex case, this case is sometimes treated, to the exclusion of the other.

Lemma 0. *X is a bound extension of its subspace Y if and only if there is no semi-norm $\|x\|_1$ on X except $\|x\|$ itself, such that $\|x\|_1 \leqq \|x\|$ for all x, and $\|y\|_1 = \|y\|$ for all y.*

Proof. If F is a Y-boundary in the unit ball of X^*, we can define $\|x\|_1 \equiv \equiv \sup\{|f(x)| : f \in F\}$. Then $\|x\|_1 \equiv \|x\|$ if and only if F is an X-boundary, yielding the direct implication. Conversely, if $\|x\|_1$ is a semi-norm as given in Lemma 0., we can take $F = \{f \in X^* : |f(x)| \leqq \|x\|_1 \text{ for all } x\}$ as a Y-boundary; if F is an X-boundary $\|x\|_1 \equiv \|x\|$.

We shall use the term *P_1 space* for a Banach space Z with the "extension property": for each Banach space X, subspace Y, and bounded linear transformation T of Y into Z, there is an extension T' of T mapping X into Z, and $\|T'\| = \|T\|$. For a discussion of P_1 spaces see KELLEY [3]. A natural and obvious example is the space $B(S)$ of all bounded function on an abstract set S, under the usual sup-norm; this example shows that every Banach space can be extended to a P_1 space. The best possible extension is discussed in the next two paragraphs.

Theorem 1. *If X is a P_1 space and Y a linear subspace of X, there is a subspace $Z \supseteqq Y$, which is a P_1 space and a bound extension of Y.*

Proof. Consider the family N of semi-norms on X subject to the conditions of Lemma 0. Zorn's Lemma yields the existence of a minimal element $\|x\|_0$ in N. X being a P_1 space there exists a linear transformation T of X into itself such that $T(y) = y$ for each y in Y and $\|T(x)\| \leqq \|x\|_0$ for each x in X. Then the semi-norm $\|x\|_1 = \|T(x)\|$, $x \in X$, belongs to N, whence $\|x\|_1$ coincides with $\|x\|_0$. Still another element of N is defined by $\|x\|_2 \equiv \limsup_n \left\| \frac{1}{n} \sum_1^n T^i(x) \right\|$. (In fact the limit exists.)

Since $\|x\|_2 \leqq \|x\|_0$ for all x, and T fixes each member of Y, $\|x\|_2 \equiv \|x\|_0$. T being a contraction of X (with respect to its original norm), it is well known that for every x, $\|(I-T)x\|_2 = 0$, or what is the same, $T^2 = T$. The space $Z = T(X)$ contains Y and is clearly a P_1 space. Finally, observe that each semi-norm $\|z\|_1'$ on Z has a natural extension to X by the definition $\|T(x)\|_1 \equiv \|x\|_1$; the minimal nature of $\|T(x)\|$ shows then that Z, with its given norm, is a bound extension of Y.

Corollary 2. *Let Z be a P_1 space which is a bound extension of a subspace Y, and U an isometry of Y into a space X such that X is a bound extension of $U(Y)$. There is a unique linear transformation V of X into Z for which*

(i) $UV(y) = y$ *for* $y \in Y$, (ii) $\|V\| \leqq 1$.

Moreover, V is an isometry.

Proof. The existence of V is assured by the hypotheses that U be isometric and Z a P_1 space. Lemma 0 shows that V is in fact an isometry of X into Z; let V_1 be another transformation with the required properties. Then there is a linear contraction S of Z such that $SV_1(x) \equiv SV(x)$; by an argument similar to that in Theorem 1, for each Z it is true that $\|z\| = \limsup \left\| \frac{1}{n} \sum_1^n S^i(z) \right\|$; then $S = 1$ and $V_1 = V$.

In the two lemmas and theorem immediately below A is a compact Hausdorff space, Y a linear subspace of $C(A)$ (real or complex), and by hypothesis there is no proper closed subset of A in which every member of Y assumes its maximum modulus. Lemmas 3 and 4 contain the irreducible kernel of analysis necessary for "concrete" applications.

Lemma 3. *If U is a non-empty open subset of A and $\frac{\pi}{4} > \varepsilon > 0$, there is an element y in Y which assumes its maximum modulus only in*

$$U \cap \{a\colon \operatorname{Re} y(a) > \|y\| \cos \varepsilon\}.$$

Proof. Let y_1 be an element of Y which assumes its maximum modulus only in U and moreover $\|y\| = \max \operatorname{Re} y_1$. Let y_2 be an element of Y which attains its maximum modulus only in the set

$$U \cap \left\{a\colon \operatorname{Re} y_1(a) > \|y\|_1 \cos \frac{\varepsilon}{3}\right\}, \quad \text{and} \quad \|y\|_2 = \max \operatorname{Re} y_2.$$

We shall show that for all sufficiently large n the functions $h_n = ny_2 + y_1$ in Y are suitable for the present lemma.

Indeed suppose $a_n \in A$ and $|h_n(a_n)| = \|h_n\|$, $n = 1, 2, 3, \ldots$. Since $\|h_n\| \geqq n\| \|y_2\| + \|y\|_1 \cos \frac{\varepsilon}{3}$ for each n, $\left(n\|y\|_2\| + \|y_1\| \cos \frac{\varepsilon}{3}\right)^2 \leqq |ny_2(a_n)|^2 + 2n \operatorname{Re}[y_2(a_n)\overline{y_1(a_n)}] + |y_1(a_n)|^2$. Thus $|y_2(a_n)| \to \|y_2\|$ and $\liminf \operatorname{Re} [y_2(a_n)\overline{y_1(a_n)}] \geqq \|y_1\|y_2\| \cdot \cos \frac{\varepsilon}{3}$. But the argument of $y_1(a_n)$ is ultimately confined to $\left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right]$ so that the argument of $y_2(a_n)$ is ultimately confined to $\left[-\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3}\right]$; the argument of $h_n(a_n)$ is then also restricted as asserted, for n sufficiently large.

Lemma 4. *Let $f \in C(A)$ and $0 \leqq d < \|f\|$. Then for some y in Y, $\|y+f\| < \|y\| - d$.*

Proof. We can assume that f attains the value $\|f\|$ in A. Let $d < s < \|f\|$ and V be a neighborhood of -1 in the plane. Then there is a function y in Y which attains its maximum modulus, 1, only in the set $\{a\colon \operatorname{Re} f(a) > s\} \cap \{a\colon y(a) \in V\}$. Once y is chosen we can estimate $\|f+ny\|$ as $n \to \infty$: $|f(a)+ny(a)|^2 = |f(a)|^2 + 2n \operatorname{Re}[f(a)\overline{y(a)}] + n^2 |y(a)|^2 \leqq n^2 + 2nB + o(n)$, where $B = \sup\{\operatorname{Re}[\bar{\lambda} f(a)]\colon \lambda \in V, a \in A, \operatorname{Re} f(a) \geqq s\}$. If the lemma is false then for every choice of s and V we must have $\|f+ny\| \geqq n-d$, whence $B \geqq -d$. Passing to the limit as V contracts to -1, we obtain an a such that $\operatorname{Re} f(a) \geqq s$, and $-\operatorname{Re} f(a) \geqq -d$, or $\operatorname{Re} f(a) \leqq s$, a contradiction proving the lemma.

Theorem 5. *$C(A)$ is a bound extension of Y.*

Proof. If $\|x\|_1$ is a semi-norm as in Lemma 0, then for every $y \in Y, f \in C(A)$, we have $\|y+f\| \geqq \|y+f\|_0 \geqq \|y\| - \|f\|_0$. By Lemma 4, $\|f\| = \|f\|_0$.

Corollary 6. *A Banach space X is a bound extension of a subspace Y if and only if the unit ball of X^* contains a w^*-closed X-boundary F which is a minimal w^*-closed Y boundary.*

Proof. The converse assertion is clear inasmuch as any Y-boundary is an X-boundary. On the other hand if the set F exists we can consider that $Y \subseteqq X \subseteqq C(F)$ and by Theorem 5, $C(F)$ is a bound extension of Y. An easy application of the Hahn—Banach Theorem shows that X, too, is a bound extension of Y.

Let us apply the previous remarks to a P_1 space X, and a minimal w^*-closed X-boundary A in the unit ball of X^*. By Theorem 5, $C(A)$ is a bound extension of the P_1 space X, whence $X = C(A)$. Again from the definition of P_1 space, there is a projection T of the Banach space of bounded functions on A onto $C(A)$, the projection T having norm one. Since $C(A)$ contains the constant functions, it is plain that T must preserve the class of non-negative real functions. In particular if h is the characteristic function of an open subset U of A, then $T(h) = 1$ on U and $T(h) = 0$ on the complement of U^-. Then U^- must be open: A is extremally disconnected (Kelley [3]).

To complete our previous considerations, and obtain incidentally a converse to the last remark, we require a lemma on regular open sets. For the necessary theory of Boolean algebras, one may consult Halmos [2], in particular pages 13—17. We adopt the notation that ϱS be the interior of the closure of S for any subset S in a topological space, and $\mathscr{R}(M)$ be the Boolean algebra of regular open subsets of M.

Lemma 7. *Let f be a continuous mapping of a compact Hausdorff space M onto a Hausdorff space N, such that $f(S) \neq N$ for any proper closed subset S of M. There is defined a Boolean isomorphism m of $\mathscr{R}(M)$ onto $\mathscr{R}(N)$:*

$$mU = \varrho f(U), \qquad U \in \mathscr{R}(M).$$

The inverse s is given by

$$sV=\varrho f^{-1}(V), \qquad V\in\mathscr{R}(N).$$

Proof. We verify first that m and s are inverse to each other. If $V\in\mathscr{R}(N)$, surely $V\subseteqq msV$; since $f^{-1}(V)$ is dense in $f^{-1}(msV)$, V is dense in msV and $V=msV$. If $U\in\mathscr{R}(M)$, then $f(smU)$ contains the interior of $f(U)$, so $f(U')\cup f(smU)$ is dense in N and $U'\cup smU$ is dense in M, yielding $U\subseteqq smU$. From the fact that $f(U)$ is dense in $f(smU)$ it follows similarly that $U\supseteqq smU$.

The identity $\varrho(E\cup F)=\varrho E\vee\varrho F$ for arbitrary subsets $E, F\subseteqq N$, shows that $m(U_1\vee U_2)=mU_1\vee mU_2$ for $U_1, U_2\in\mathscr{R}(M)$; this depends on the fact that $U_1\cup U_2$ is dense in $U_1\vee U_2$. To see that $m(U_1\cap U_2)=mU_1\cap mU_2$, observe that for certain open subsets W_1 and W_2 of N, $f^{-1}(W_i)\subseteqq U_i$ and $f^{-1}(W_i)$ is dense in U_i, $i=1,2$. Then $f^{-1}(W_1\cap W_2)$ is dense in $U_1\cap U_2$, and $m(U_1\cap U_2)=\varrho(W_1\cap W_2)=$ $=\varrho W_1\cap\varrho W_2=mU_1\cap mU_2$ (lemma 4, p. 15, [2]). The facts now established for m and s complete the proof.

Corollary 8. *If N is extremally disconnected, f is a homeomorphism* (GLEASON [1; Lemma 2.3]).

Proof. Since $\mathscr{R}(N)$ contains only closed subsets of N, $\mathscr{R}(M)$ contains only subsets of the form $f^{-1}(W)$ for a subset W open in N; the same form prevails for all open subsets of M, whence f is one-to-one.

Returning to the general problem, we begin with a Banach space X and a minimal w^*-closed X-boundary F in the unit ball of X^*. Also, let Z be a bound P_1 extension of X; we know at the outset that Z is isometric with $C(A)$ for some extremally disconnected compact Hausdorff space A. We shall show that A is the Stone space of the Boolean algebra $\mathscr{R}(F)$ and that $\mathscr{R}(F)$ is independent (to within isomorphism) of the choice of F.

The first step is to consider a w^*-closed subset F_1 of the unit ball of Z^* whose restriction to X is exactly F; since Z is a bound extension of X, F_1 is a Z-boundary. Since F is a minimal closed X-boundary, we can suppose that F_1 is a minimal closed Z-boundary. If we use the familiar representation of Z^* by countably additive Borel measures in A, we see that F_1 must contain, for each $a\in A$, a measure with mass 1 at $\{a\}$; this does not depend on the disconnectedness of A. It is convenient to write $\lambda\cdot a$ for the functional $f\to\lambda f(a)$, $f\in C(A)$, $a\in A$, λ a complex number. The measures in F_1 which can be represented in the manner just described form a closed subset which is a boundary for Z and consequently they exhaust F_1, by the minimality. The mapping π of F_1 onto A given by $\pi(\lambda\cdot a)=a$ for $\lambda\cdot a$ in F_1, is continuous and fulfills the conditions of Lemma 7 and Corollary 8, insuring that π is a homeomorphism of F_1 onto A. The Boolean algebra $\mathscr{R}(F_1)$ (or $\mathscr{R}(A)$) determines A to within homeomorphism, while $\mathscr{R}(F_1)\cong\mathscr{R}(F)$ by Lemma 7. This is the conclusion sought, in view of the fact that $\mathscr{R}(A)$ coincides with the Boolean algebra of open-closed subsets of A.

References

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