On interpolation functions

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If X is a locally compact space provided with a positive measure μ, we denote by L^p_ζ, where $1 < p < \infty$ and ζ is a positive μ-measurable function, the space of μ-measurable functions a such that ζa is of μ-integrable pth power. We provide L^p_ζ with the norm

$$\|a\|_{L^p_\zeta} = \Big(\int_X |\zeta a|^p \, d\mu\Big)^{1/p}.$$

A function $H = H(z_0, z_1)$, defined, Borel measurable, and positive for $z_0 > 0$, $z_1 > 0$, is said to be an *interpolation function* of power p if and only if whenever π is a linear mapping which is continuous from $L^p_{\zeta_0}$ into $L^p_{\zeta_0}$ and from $L^p_{\zeta_1}$ into $L^p_{\zeta_1}$ then π is also continuous from $L^p_{H(\zeta_0, \zeta_1)}$ into $L^p_{H(\zeta_0, \zeta_1)}$; it is understood that the domain of π contains both $L^p_{\zeta_0}$ and $L^p_{\zeta_1}$. We require also that

(1) $$M \leqq C \max\{M_0, M_1\}$$

for some constant C, where M_0, M_1, M are the corresponding operator norms, $M_0 = \sup\{\|\pi a\|_{L^p_{\zeta_0}} / \|a\|_{L^p_{\zeta_0}}\}$ etc. (It is intended that this should hold for all X, μ, ζ_0, ζ_1, in particular C should only depend on H.) If $C = 1$ we say (following DONOGHUE [1]) that H is an *exact interpolation function.* E.g. $z_0^{1-\theta} z_1^\theta$ with $0 < \theta < 1$ is an exact interpolation function by the well-known theorem of STEIN and WEISS [4]. The first general criterion for a function to be an exact interpolation function was given by FOIAŞ and LIONS [2]. In [3] we gave a somewhat novel deduction of their condition and supplemented it by a new condition in a sense dual to the first one.

For technical reasons mainly we shall restrict below the notion of interpolation function further: We shall require that $H(z_0, z_1)$ is homogeneous of degree 1 and moreover that

(2) $$\lim_{z_0 \to 0} H(z_0, z_1) = 0, \qquad \lim_{z_1 \to 0} H(z_0, z_1) = 0.$$

We shall give necessary and sufficient conditions for a function to be a (not necessarily exact) interpolation function in the above restricted sense. It will be clear that this settles the problem of interpolation functions for most practical purposes. We note however that within the narrower class of *exact* interpolation function the question is still open, except when $p = 2$ [1], [2], but this is now of mostly theoretical interest only.

Let us say that two positive functions u and v defined in any set are *equivalent* if there exists a constant C, $0 < C < \infty$, such that $u \leqq Cv$ and $v \leqq Cu$ in that set.

It is clear that if H is an interpolation function then every equivalent function (homogeneous of degree 1) is also an interpolation function. Indeed, this follows from the fact that L^p_ζ is not changed if we replace ζ by an equivalent function.

Theorem 1. *Either of the two following conditions is necessary and sufficient for a function* $H=H(z_0, z_1)$ (homogeneous of degree 1) *to be an interpolation function of power* p:

i) *H is equivalent to a function of the form*

$$H_1(z_0, z_1) = z_0\left[\varphi\left(\left(\frac{z_1}{z_0}\right)^{-q}\right)\right]^{-\frac{1}{q}} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{3}$$

where φ is positive and concave and $\varphi(\sigma)=o(\max(1,\sigma))$ *as* $\sigma\to 0$ *or* ∞;

ii) *H is equivalent to a function of the form*

$$H_2(z_0, z_1) = z_0\left[\psi\left(\left(\frac{z_1}{z_0}\right)^{p}\right)\right]^{\frac{1}{p}} \tag{4}$$

where ψ is positive and concave and $\psi(\sigma)=o(\max(1,\sigma))$ *as* $\sigma\to 0$ *or* ∞.

Example. If $H(z_0, z_1)=z_0^{1-\theta} z_1^{\theta}$ with $0<\theta<1$ we may take $\varphi(t)=\psi(t)=t^{\theta}$ and we obtain thus a weak form of the theorem of Stein and Weiss [4].

Proof (sufficiency). Recall first the result of [3]: A function $H=H(z_0, z_1)$ is an exact interpolation function if it is of the form

$$H(z_0, z_1) = H_3(z_0, z_1) = \left[\int_0^\infty (z_0^p + t^p z_1^p)^{-\frac{q}{p}}\, d\xi(t)\right]^{-\frac{1}{q}} \tag{5}$$

or of the form

$$H(z_0, z_1) = H_4(z_0, z_1) = \left[\int_0^\infty (z_0^{-q}+t^{-q} z_1^{-q})^{-\frac{p}{q}}\, d\eta(t)\right]^{\frac{1}{p}} \tag{6}$$

where $\xi(t)$ and $\eta(t)$ are increasing functions and $\frac{1}{p}+\frac{1}{q}=1$. (In [3] it was assumed that $\xi(t)$ and $\eta(t)$ were absolutely continuous but this is of course immaterial.) We note that in either case (2) is automatically fulfilled.

Let us first consider the case of ii). It is plain that any H_4 is equivalent to a function of the form

$$z_0\left(\int_0^\infty \min\left\{1, \left(\frac{t z_1}{z_0}\right)^p\right\} d\eta(t)\right)^{\frac{1}{p}},$$

because $(z_0^{-q}+t^{-q}z_1^{-q})^{-\frac{1}{q}}$ and $\min(z_0, tz_1)$ are equivalent, and conversely any such a function is equivalent to a H_4 and thus by [3] it is itself and interpolation function. But since $\min\left\{1, \left(\frac{tz_1}{z_0}\right)^p\right\}$ is concave in $\left(\frac{z_1}{z_0}\right)^p$ this is a H_2 by superposition. Hence, after a change of variable, we have to show that every positive concave function $\psi = \psi(\sigma)$ with $\psi(\sigma) = o\,(\max(1, \sigma))$ as $\sigma \to 0$ or ∞ can be represented in the form

$$\psi(\sigma) = \int_0^\infty \min\{\sigma, \tau\}\, dw(\tau) \tag{7}$$

where $w(\tau)$ is increasing. Starting with formula (7) we obtain easily by integration by parts

$$\psi(\sigma) = \sigma w(\infty) - \int_0^\sigma w(\tau)\, d\tau$$

or by differentiation (at continuity points of w)

$$\psi'(\sigma) = w(\infty) - w(\sigma).$$

Thus, if we normalize w by $w(\infty) = 0$, we have

$$w(\sigma) = -\psi'(\sigma). \tag{8}$$

Conversely, if ψ is given defining w by (8) we easily obtain the desired representation (7). Indeed since $\psi(\sigma) = o\,(1)$ as $\sigma \to 0$ we have

$$\psi(\sigma) = \int_0^\sigma \psi'(\tau)\, d\tau$$

from which follows by integration by parts

$$\psi(\sigma) = \sigma\psi'(\sigma) + \int_0^\sigma \tau\, dw(\tau).$$

But since $\psi(\sigma) = o(\sigma)$ as $\sigma \to \infty$ we have also $\psi'(\sigma) = o(1)$ as $\sigma \to \infty$. So

$$\psi'(\sigma) = \int_\sigma^\infty dw(\tau)$$

and (7) follows. This settles the case of ii).

It remains the case of i). Now it is immediate that any H_3 is equivalent to a function of the form

$$z_0 \left(\int_0^\infty \min\left\{1, \left(\frac{tz_1}{z_0}\right)^{-q}\right\} d\xi(t) \right)^{-\frac{1}{q}}$$

which clearly is a H_1. The converse follows at once by adapting the above argument.

P r o o f (necessity). We base our argument on an idea taken over from DONOGHUE [1] ($p = 2$).

Assume that the space X is discrete and contains exactly $(n+1)$ points $x, x_1, \dots, x_n$ each of these points carrying the mass 1. We take $\zeta_0 = 1$ and $\zeta_1(x) = z$, $\zeta_1(x_i) = z_i$ $(i = 1, \dots, n)$ where $\left(\frac{1}{z}\right)^q$ is assumed to be in the convex closure of $\left(\frac{1}{z_i}\right)^q$ $(i = 1, \dots, n)$. We define a linear mapping π_1 by setting

$$\begin{cases} (\pi_1 a)(x) = \alpha_1 a(x_1) + \dots + \alpha_n a(x_n) \\ (\pi_1 a)(x_i) = 0 \qquad (i = 1, \dots, n) \end{cases}$$

where $\alpha_1, \dots, \alpha_n$ are to be determined. The norm of π_1, as a mapping from $L^p_{\zeta_0}$ into $L^p_{\zeta_0}$, is

$$M_0 = \sup_a \left\{ |\alpha_1 a(x_1) + \dots + \alpha_n a(x_n)| / (|a(x_1)|^p + \dots + |a(x_n)|^p)^{\frac{1}{p}} \right\} = (\alpha_1^q + \dots + \alpha_n^q)^{\frac{1}{q}}$$

or, as a mapping from $L^p_{\zeta_1}$ into $L^p_{\zeta_1}$,

$$M_1 = \sup_a \left\{ z |\alpha_1 a(x_1) + \dots + \alpha_n a(x_n)| / (|z_1 a(x_1)|^p + \dots + |z_n a(x_n)|^p)^{\frac{1}{p}} \right\}$$

$$= z \left(\left(\frac{\alpha_1}{z_1}\right)^q + \dots + \left(\frac{\alpha_n}{z_n}\right)^q \right)^{\frac{1}{q}},$$

this is an immediate consequence of HÖLDER's inequality. We choose now $\alpha_1, \dots, \alpha_n$ such that $M_0 = M_1 = 1$, i. e.

$$\alpha_1^q + \dots + \alpha_n^q = 1, \qquad \left(\frac{\alpha_1}{z_1}\right)^q + \dots + \left(\frac{\alpha_n}{z_n}\right)^q = \left(\frac{1}{z}\right)^q,$$

which is possible since $\left(\frac{1}{z}\right)^q$ is in the convex closure of $\left(\frac{1}{z_i}\right)^q$ $(i = 1, \dots, n)$. But the norm of π_1 as a mapping from $L^p_{H(\zeta_0, \zeta_1)}$ into $L^p_{H(\zeta_0, \zeta_1)}$ is

$$M = H(1, z) \left[\left(\frac{\alpha_1}{H(1, z_1)} \right)^q + \dots + \left(\frac{\alpha_n}{H(1, z_n)} \right)^q \right]^{\frac{1}{q}}.$$

So if $H(z_0, z_1)$ is an interpolation function we get from (1)

$$\left(\frac{\alpha_1}{H(1, z_1)} \right)^q + \dots + \left(\frac{\alpha_n}{H(1, z_n)} \right)^q \leq C \left(\frac{1}{H(1, z)} \right)^q.$$

Thus writing

$$H(1, z) = [\varphi(z^{-q})]^{-\frac{1}{q}}, \quad \sigma = z^{-q}, \quad \sigma_i = z_i^{-q}, \quad \lambda_i = \alpha_i^q \qquad (i = 1, \dots, n)$$

we get

$$(9) \qquad \begin{cases} \lambda_1 + \dots + \lambda_n = 1, \\ \lambda_1 \sigma_1 + \dots + \lambda_n \sigma_n = \sigma, \\ \lambda_1 \varphi(\sigma_1) + \dots + \lambda_n \varphi(\sigma_n) \leq C \varphi(\sigma) \end{cases}$$

for any $\sigma, \sigma_1, \ldots, \sigma_n$ with σ in the convex closure of $\sigma_1, \ldots, \sigma_n$. It follows now readily that φ is equivalent to a concave function. (If φ satisfies (9) then $\varphi^*(\sigma) = \sup \{\lambda_1\varphi(\sigma_1) + \ldots + \lambda_n\varphi(\sigma_n)\}$ is concave, i.e. satisfies (9) with $C=1$.) Thus we have shown that every interpolation function is equivalent to a function of the form (3) so i) is a necessary condition.

The necessity of (4) can be proven in a similar fashion by using a linear mapping π_2 defined by a "dual" condition

$$(\pi_2 a)(x) = 0, \qquad (\pi_2 a)(x_i) = \beta_i a(x_i) \qquad (i = 1, \ldots, n)$$

where $\beta_1, \ldots, \beta_n$ are to be determined. We leave the details to the reader.

The proof of theorem 1 is complete.

By making vary p we obtain from theorem 1 the following somewhat surprising byproduct.

Corollary. *Let $\chi = \chi(\sigma)$ be a positive concave function with $\chi(\sigma) = o(\max\{1, \sigma\})$ as $\sigma \to 0$ or ∞. Then for any real number $r \neq 0$ the function $\chi^*(\sigma) = (\chi(\sigma^r))^{1/r}$ is equivalent to a concave function.*

(χ^* may itself not concave (in general), unless $0 < r < 1$.)

Strangely enough we have not been able to give a direct proof of this corollary which does not contain at least some idea from the theory of interpolation spaces.

In the light of the above corollary we can restate theorem 1 in the following more general form:

Theorem 1′. *A function $H = H(z_0, z_1)$ is an interpolation function of power p if and only for some $r \neq 0$ it is equivalent to a function of the form*

$$z_0 \left[\chi\left(\left(\frac{z_1}{z_0} \right)^r \right) \right]^{\frac{1}{r}}$$

where χ is positive and concave and $\chi(\sigma) = o(\max\{1, \sigma\})$ as $\sigma \to 0$ or ∞.

It follows in particular that if a function is an interpolation function of power p for some p then it is so for all p, $1 < p < \infty$.

We conclude by the following

Remark. It is easy to see that if $F(z_0, z_1)$ is an interpolation function of power p then $1/F\left(\frac{1}{z_0}, \frac{1}{z_1}\right)$ is an interpolation function of power q and vice versa. (We owe this observation to J. L. Lions.) In particular we see that (5) and (6) follow directly from each other. Using this circumstance the above proof can be somewhat simplified.

References

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