On some results of A. Rényi and C. Rényi concerning periodic entire functions

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In their paper [7] A. and C. RÉNYI have investigated the possibility that $f(g(z))$ should be an entire periodic function, where $f(z)$ and $g(z)$ are entire and $g(z)$ is non-periodic. They proved the following two theorems:

Theorem 1. *If $f(z)$ is an arbitrary non-constant entire function and $P(z)$ an arbitrary polynomial of degree $\geqq 3$, then the entire function $f(P(z))$ is not periodic.*

Theorem 2. *If $P(z)$ is an arbitrary non-constant polynomial and $g(z)$ an entire function which is not periodic, then $P(g(z))$ is not periodic.*

In this note we shall make some improvements on theorem 2 and prove first

Theorem 3. *If $f(z)$ is an arbitrary non-constant entire function of order less than $\frac{1}{2}$ or of order $\frac{1}{2}$ and minimal type, and if $g(z)$ is an entire function which is not periodic, then $f(g(z))$ is not periodic.*

The proof depends on the following

Lemma 1. *If $f(z)$ and $g(z)$ are non-constant entire functions such that $g(z)$ is not periodic but $f(g(z))=F(z)$ is periodic with period λ, and if t is a value for which $F'(t)\neq 0$, then for integral n, $w_n=g(t+n\lambda)$ satisfies $f(w_n)=F(t)$, while $w_n=w_m$ if and only if $n=m$.*

Consequently $|g(t+n\lambda)|\to\infty$ as $|n|\to\infty$ and $g(z)$ is unbounded on any ray $z=z_0+\lambda s$, $0\leqq s<\infty$.

Proof of Lemma 1. It is clear that w_n satisfies $f(w_n)=f(g(t+n\lambda))=$ $=F(t+n\lambda)=F(t)=a$, say. If there are integers m, n such that $m\neq n$, $w_m=w_n$, then consideration of

$$a(s)=F(t+m\lambda+s)=f(g(t+m\lambda+s))=F(t+n\lambda+s)=f(g(t+n\lambda+s))$$

shows that for all sufficiently small $|s|$,

$$g(t+m\lambda+s)=g(t+n\lambda+s)$$

is the unique solution of $f(w)=a(s)$ near $w_m=w_n$ (we recall that $f'(w_m)\neq 0$ since $f'(w_m)g'(t+m\lambda)=F'(t+m\lambda)=F'(t)\neq 0$). It then follows that $g(z)$ is periodic with period $(m-n)\lambda$, against assumption. Thus we conclude that $w_m\neq w_n$ for $m\neq n$.

Since w_n is a solution of $f(w_n)=a$, it follows that $|w_n|\to\infty$ as $|n|\to\infty$. On any ray L: $z=z_0+\lambda s$, $s\geqq 0$, one can find $z=t$ such that $F'(t)\neq 0$, and the $t+n\lambda$ then lie on L so that $g(z)$ is unbounded on L.

Proof of theorem 3. Suppose that f and g satisfy the conditions of theorem 3, but that $F(z)=f(g(z))$ is periodic with period, say, λ. Denote by L any ray of the form $z=z_0+\lambda s$, $s\geqq 0$. Now $g(L)$ is an unbounded, connected plane set on which $f(z)$ is bounded, since the values of $f(z)$ on $g(L)$ are the values of $F(z)=f(g(z))$ on L, and these are bounded, being in fact the values taken on the intersection of L with one period strip of F. But (see e.g. [3]) for a function of the order of growth of $f(z)$ there is a sequence $R_n\to\infty$ such that the minimum of $|f(z)|$ tends to ∞ as $|z|\to\infty$ through the values R_n. This contradicts the boundedness of $f(z)$ on $g(L)$. Thus the original assumption that $f(g(z))$ is periodic is false.

Theorem 3 is sharp, since to a prescribed type $\varepsilon>0$ the function $f(z)=\cosh(\varepsilon\sqrt{z})$ is of order $\frac{1}{2}$, type ε and $f(z^2)=\cosh(\varepsilon z)$ is periodic. We may note, however, that in this example the "inner" function $g(z)$ is the polynomial z^2 and it is natural to see if more can be proved when the case of quadratic $g(z)$ is set aside. Concerning this case one can at least prove

Theorem 4. *If $f(z)$ is an arbitrary non-constant entire function, if $g(z)$ is a non-periodic entire function other than a polynomial of degree $\leqq 2$, and if $F(z)=f(g(z))$ is periodic, then*

(i) *$g(z)$ is transcendental, and* (ii) *the order of $F(z)$ is infinite.*

Proof. Part (i) follows at once from Theorem 1. The proof of part (ii) follows at once from the result of Pólya [6] that if $F=f(g)$ is of finite order, where f, g are entire, then either g is a polynomial and f is of finite order or g is not a polynomial and f is of zero order. Since in our case g is not a polynomial and f has order $\geqq\frac{1}{2}$ we conclude that $F(z)$ has infinite order.

For the further discussion we note that without loss of generality z may be subjected to a linear transformation to make the period of $F(z)$ equal to $2\pi i$, i.e. $F(z)$ may be represented as $h(e^z)$, where $h(z)=\sum_{-\infty}^{\infty}A_nz^n$, $0<|z|<\infty$. The case when $h(z)$ is entire, i.e. when $A_n=0$ for negative n, may be distinguished as the case when $F=h(e^z)$ is bounded in the left half-plane $\operatorname{Re} z<0$. In this case it is impossible that a non-constant $F=f(g(z))$, where $g(z)$ is a quadratic polynomial, for such a decomposition together with the boundedness of F in a left half-plane would imply boundedness in a right half-plane also and thus F would be a (periodic) entire function bounded in the whole plane. We can prove rather more:

Theorem 5. *If $h(z)$ is a non-constant entire function and $F(z)=h(e^z)$, and if F may also be represented as $F=f(g(z))$, where $g(z)$ is a non-periodic entire function and $f(z)$ is an entire function, then*

(i) *the order of $f(z)$ is at least one,*
(ii) *the order of $g(z)$ is at least one,*
(iii) *$g(z)$ is p-valent in a suitable left half-plane H:* $\operatorname{Re} z<\text{const.}$ *for some integer p, i.e. g takes any value at most p times in H,*
(iv) *the order of $F(z)$ is infinite, so that $h(z)$ must be transcendental.*

In the course of this proof we shall need two lemmas:

Lemma 2. *If for a positive integer p an entire function g(z) is p-valent in a half-plane H:* $\operatorname{Re} z < C = \text{const.}$ *in the sense that g(z) takes no value more than p times in H, then*

$$|g(z)| = O(|z|^{2p})$$

uniformly as $z \to \infty$ *in any angle A:* $|\arg z - \pi| \leqq \theta$, *where* θ *is a constant less than* $\pi/2$.

Such a result was proved by BIEBERBACH [2] in the case $p = 1$ of univalent functions.

Proof of Lemma 2. Put $t = z - C$, $s = (t+1)/(t-1)$. This substitution maps H one-to-one conformally on D: $|s| < 1$ in such a way that $z = \infty$ corresponds to $t = 1$. Moreover $\varphi(s) \equiv g(z)$ is p-valent in D. By Miss CARTWRIGHT's results [4] one has

$$|\varphi(s)| = O\{(1-|s|)^{-2p}\} \quad \text{uniformly as} \quad |s| \to 1 \quad \text{in } D.$$

Now, as $z \to \infty$ in an angle A, so that for large $|z|$, z is in H, we have

$$1 - |s|^2 = 1 - \frac{t+1}{t-1} \cdot \frac{\bar{t}+1}{\bar{t}-1} = -2 \operatorname{Re} t/(t\bar{t} - t - \bar{t} + 1)$$

$$> -2 \operatorname{Re} t/(|t|+1)^2 \quad \text{since} \quad \operatorname{Re} t < 0,$$

$$> K|z| \cos\theta/|z|^2 = K'/|z| \quad \text{for a suitable constant } K'.$$

Thus $|g(z)| = |\varphi(s)| = O\{(1-|s|)^{-2p}\} = O\{(1-|s|^2)^{-2p}\} = O(|z|^{2p})$.

The next lemma is essentially the Phragmén—Lindelöf principle in a form convenient for our application. It is proved in this form in e.g. [1].

Lemma 3. *If the order of the entire function f(z) is* $\leqq \beta$, $\beta > 0$, *and if as* $z \to \infty$ *outside a number of disjoint angular sectors of the form D:*

$$\theta_1 < \arg z < \theta_2, \quad \theta_2 - \theta_1 < \frac{\pi}{\beta},$$

one has

$$|f(z)| = O(\exp(|z|^{\beta'})), \quad \beta' < \beta, \quad K \ \textit{constant},$$

then the order of f(z) is in fact $\leqq \beta'$.

Taking $\beta < 1$ one immediately obtains the

Corollary. *An entire function of order* $\beta < 1$ *cannot be of order strictly* $< \beta$ *(in particular cannot be* $O(|z|^k)$*) in any half-plane.*

Proof of theorem 5. If $h(0) = \alpha$, then $F(z) \to \alpha$ as $\operatorname{Re} z \to -\infty$.

First we show $|g(x)| \to \infty$ as $x \to -\infty$. Suppose this is not true. Then there is a $K > 0$ and a sequence $x_n \to -\infty$, such that $|g(x_n)| \leqq K$. Now $F(x) = f(g(x)) \to \alpha$ as $x \to -\infty$. We can assume, by choosing a subsequence of x_n if necessary, that $g(x_n) \to \beta$, $|\beta| \leqq K$. Then $f(\beta) = \alpha$. Now given $\varepsilon > 0$, we have $|f(g(x)) - \alpha| < \varepsilon$ for all sufficiently large $-x$, and since the α-points of f are isolated, it follows that $g(x) \to \beta$ as $x \to -\infty$. Indeed $f(g(z)) \to \alpha$ as $\operatorname{Re} z \to -\infty$, so that we must have $g(z) \to \beta$ as

Re $z \to -\infty$. But this is impossible, since by Lemma 1 $g(z)$ is unbounded on any line Re z=constant. Thus we have shown that $|g(x)| \to \infty$ as $x \to -\infty$. Consider the half-plane H: Re $z < c$, where c is chosen so that $h(z)$ takes the value α in $|z| < e^c$ only at $z=0$, while $|h(z)-\alpha| > \delta > 0$ on $|z| = e^c$. Then in H one has $F(z) \neq \alpha$, while on the boundary of H one has $|F(z)-\alpha| > \delta > 0$. As $z \to \infty$ on the negative real axis R in H one has $F(z) \to \alpha$.

The values of $w = g(z)$ in H form an unbounded domain $G = g(H)$ and as $z \to \infty$ along R in H, the corresponding values $g(z)$ run to ∞ along a path L in G. As $w \to \infty$ along L one has $f(w) \to \alpha$. Moreover $f(w) \neq \alpha$ for w in G, while at any boundary point of G, $|f(w)-\alpha| > \delta$. Consequently (see e.g. [5, Chapter XI]) α is a direct critical transcendental singularity of the inverse function $f_{-1}(z)$ of $f(w)$. By the Denjoy—Carleman—Ahlfors theorem [5, p. 313] it follows that the order of $f(z)$ is at least one. This proves part (i).

Let the integer p denote the multiplicity of $z=0$ as a solution of $h(z) = \alpha$. Then the number $\varepsilon > 0$ and the c in the definition of H may be chosen so that, for $0 < |\alpha' - \alpha| < \varepsilon$ the equation $h(t) = \alpha'$ has exactly p roots, all different, $t_1, t_2, \ldots, t_p$ in $|t| < e^c$. Thus in H we have $F(z) = h(e^z) = \alpha'$ precisely at the infinite set of points

$$S(\alpha') = \{\log t_1, \log t_2, \ldots, \log t_p\}.$$

We may assume that $F'(z) \neq 0$ in H. Then by Lemma 1 $g(z)$ does not take the same value at any two different values of $\log t_i$ (for fixed i). Then on the set $S(\alpha)$ the function $g(z)$ can take a given value at most p times, i.e. once on a value of $\log t_1$, once on a value of $\log t_2$, etc.

Now if $g(z) = g(z')$ for z, z' in H, then $F(z) = F(z')$ so that z and z' belong to the same set $S(\alpha')$. Hence we have proved (iii) that $g(z)$ takes any given value at most p times in H.

By Lemma 2 we see that $g(z)$ is $O(|z|^{2p})$ in H and it follows from Lemma 3 that the order of $g(z)$ is at least one.

The infinite order of $F(z)$ follows from Pólya's result just as in Theorem 3.

The example $F = \exp(z + e^z)$, $f = e^z$, $g = z + e^z$, $h = ze^z$ shows that order 1 can indeed occur for both f and g. In this case $p = 1$.

References

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