A note on operators whose spectrum is a spectral set

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The reader is referred to [6] and [7] for terminology, and for the basic properties of spectral sets. If A is an operator we write $\sigma(A)$ for the spectrum of A, and $r(A)$ for the spectral radius of A. If S is a compact set of complex numbers, and f is an S-analytic function, we write $\|f\|_S = \sup\{|f(\lambda)|: \lambda \in S\}$. Thus, to say that S is a spectral set for A means: (i) $\sigma(A) \subset S$, and (ii) $\|f(A)\| \leqq \|f\|_S$ for every rational function f with no poles in S.

A key result proved in [6] is the following (VON NEUMANN's Spectral Mapping Theorem): *If S is a spectral set for the operator A and f is any S-analytic function, then $f(S)$ is a spectral set for the operator $f(A)$* ([6, p. 226, 3.4 (ii)]; see [2] for a recent exposition). It is implicit in VON NEUMANN's theorem that $\sigma[f(A)] \subset f(S)$. Moreover, C. FOIAŞ has shown that the "spectral mapping formula" holds:

$$\sigma[f(A)] = f[\sigma(A)], \tag{1}$$

where f is any S-analytic function, S being a spectral set for A [4, p. 369, (i)]. The first aim of this note is to present an elementary proof of (1) in the special case that $S = \sigma(A)$ (of course this places a restriction on the operator A):

Theorem 1. *If $\sigma(A)$ is a spectral set for the operator A, then* (1) *holds for every $\sigma(A)$-analytic function f.*

The proof is based on a general lemma:

Lemma. *If S is a spectral set for the operator A, then*

$$f[\sigma(A)] \subset \sigma[f(A)] \subset f(S) \tag{2}$$

for every S-analytic function f.

Proof. Assuming $\lambda \in \sigma(A)$, let us show $f(\lambda) \in \sigma[f(A)]$. Let f_n be a sequence of rational functions with no poles in S, such that $f_n \to f$ uniformly on S. Then $f_n(A) \to f(A)$ in norm [6, p. 264, 3.3. (I)]; since also $f_n(\lambda) \to f(\lambda)$, we have

$$f_n(A) - f_n(\lambda)I \to f(A) - f(\lambda)I \quad \text{in norm.}$$

By the spectral mapping formula for rational functions (which is no deeper than the spectral mapping formula for polynomial functions), we have

$$f_n(\lambda) \in f_n[\sigma(A)] = \sigma[f_n(A)];$$

thus the operators $f_n(A) - f_n(\lambda)I$ are singular, hence $f(A) - f(\lambda)I$ is also singular.

Proof of Theorem 1. Put $S=\sigma(A)$ in (2).

Corollary 1. *If $\sigma(A)$ is a spectral set for the operator A and f is any $\sigma(A)$-analytic function, then $\sigma[f(A)]$ is a spectral set for $f(A)$.*

Proof. By VON NEUMANN's theorem, $f[\sigma(A)]$ is a spectral set for $f(A)$; cite formula (1).

An operator A is called *normaloid* if

$$\|A\|=\sup\{|Ax, x)|: \|x\|=1\};$$

this is equivalent to the condition $r(A)=\|A\|$ by an elementary argument [3, proof of Theorem 3].

Corollary 2. *If $\sigma(A)$ is a spectral set for the operator A and f is any $\sigma(A)$-analytic function, then $f(A)$ is normaloid.*

Proof. Since the function $u(\lambda)\equiv\lambda$ is $\sigma(A)$-analytic, we have $\|A\|=\|u(A)\|\leqq \leqq\|u\|_{\sigma(A)}=r(A)$; but $r(A)\leqq\|A\|$ (for any operator), thus $r(A)=\|A\|$, and so A is normaloid. By Corollary 1, the same argument is applicable to $f(A)$, thus $f(A)$ is normaloid.

The special case of Corollary 2 for rational functions f with no poles in $\sigma(A)$ was proved by S. HILDEBRANDT [5, p. 421, Corollary]. The following result, related to Corollary 2, is much more elementary; it is implicit in [5, p. 420, Remark]

Theorem 2. *In order that $\sigma(A)$ be a spectral set for the operator A, it is necessary and sufficient that $f(A)$ be normaloid for every rational function f with no poles in $\sigma(A)$.*

Proof. If f is a rational function with no poles in $\sigma(A)$, then $\sigma[f(A)]=f[\sigma(A)]$ by elementary considerations, and so $r[f(A)]=\|f\|_{\sigma(A)}$.

Suppose first that $\sigma(A)$ is a spectral set for A. If f is any rational function with no poles in $\sigma(A)$, then $\|f(A)\|\leqq\|f\|_{\sigma(A)}=r[f(A)]\leqq\|f(A)\|$, thus $f(A)$ is normaloid.

Conversely, if $f(A)$ is normaloid for all rational functions f with no poles in $\sigma(A)$, then $\|f(A)\|=r[f(A)]=\|f\|_{\sigma(A)}$ for all such f, thus $\sigma(A)$ is a spectral set for A

An operator A is called *hyponormal* if $AA^*\leqq A^*A$.

Corollary 1. *If A is an operator such that $f(A)$ is hyponormal for all rational functions f with no poles in $\sigma(A)$, then $\sigma(A)$ is a spectral set for A.*

Proof. It suffices to cite the theorem, due to T. ANDÔ [1], that a hyponormal operator is normaloid. Incidentally, here is an elementary proof of ANDÔ's theorem that avoids any reference to spectrum: if A is hyponormal, then $\|A^n\|=\|A\|^n$ for all positive integers n [8, proof of Theorem 1], and so A is normaloid [3, proof of Theorem 2].

Corollary 2. (VON NEUMANN) *If A is a normal operator, then $\sigma(A)$ is a spectral set for A.*

Proof. Since $f(A)$ is obviously normal for every rational function f with no poles in $\sigma(A)$, the assertion is immediate from Corollary 1. We remark that the proof does not use the spectral theorem (*cf.* [6, p. 277]).

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