On the strong summability of orthogonal series

By LÁSZLÓ LEINDLER in Szeged

1. Let $\{\varphi_n(x)\}$ $(n=0, 1, \ldots)$ be an orthonormal system on the interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, the series (1. 1) converges in the mean to a square-integrable function $f(x)$. By $s_n(x)$ and $\sigma_n^\alpha(x)$ we denote the n-th partial sums and the n-th Cesàro means of order $\alpha(>-1)$ of the series (1. 1), i.e.

$$s_n(x) = \sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x)$$

and

$$\sigma_n^\alpha(x) = \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} s_\nu(x) \qquad \left(A_n^{(\alpha)} = \binom{n+\alpha}{n}\right).$$

2. Concerning the strong and very strong summability of (1. 1), SUNOUCHI [3] proved recently the following theorems:

Theorem A. *If the orthogonal series* (1. 1) *with* (1. 2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b), *then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_\nu(x)-f(x)|^k = 0$$

almost everywhere in (a, b) *for any* $\alpha>0$ *and* $k>0$.

Theorem B. *If*

$$\sum_{n=4}^{\infty} c_n^2 (\log\log n)^2 < \infty, \tag{2.1}$$

then

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_{k_\nu}(x)-f(x)|^k = 0$$

holds for any $\alpha>0$ *and* $k>0$, *almost everywhere in* (a, b), *for any increasing sequence* $\{k_\nu\}$.

TANDORI [4] has proved this theorem for $\alpha=1$ earlier.

In [2] we have generalized this theorem of TANDORI as follows:

Theorem C. *Under the hypothesis* (2. 1) *we have*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}[s_{l_\nu}(x)-f(x)]^2=0$$

almost everywhere for any (non necessarily monotonic) sequence $\{l_\nu\}$ *of distinct non-negative integers.*

At the same time we proved the following

Theorem D. *Let* $\{a_n\}$ *be a given sequence of real numbers with* $\sum a_n^2<\infty$ *and*

$$na_n^2\geqq(n+1)a_{n+1}^2\qquad(n=1,2,\ldots).$$

If the orthogonal series (1. 1) *with* (1. 2) *is Abel-summable to* $f(x)$ *almost everywhere in* (a, b) *and*

$$c_n^2=O(a_n^2),$$

then we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}[\sigma_{l_\nu}^{\gamma-1}(x)-f(x)]^2=0$$

for any $\gamma>\frac{1}{2}$ *almost everywhere in* (a, b), *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

3. In the present note we intend to generalize further these theorems.

We consider a regular summation method T_n determined by a triangular matrix $\left\|\frac{\alpha_{nk}}{A_n}\right\|$ $\left(\alpha_{nk}\geqq0 \text{ and } A_n=\sum_{k=0}^{n}\alpha_{nk}\right)$, i.e. if s_k tends to s, then

$$T_n=\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}s_k\to s.$$

Theorem 1. *Let* $k>0$. *If there exists a* $p>1$ *such that*

(3. 1) $$\frac{p}{p-1}k\geqq2\quad\text{and}\quad\left\{\sum_{\nu=1}^{n}\nu^{p-1}\alpha_{n\nu}^{p}\right\}^{1/p}\leqq K\sum_{\nu=1}^{n}\alpha_{n\nu}$$

and if the series (1. 1) *with* (1. 2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b), *then*

(3. 2) $$\lim_{n\to\infty}\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-f(x)|^k=0$$

almost everywhere in (a, b) *for any* $\gamma>\frac{1}{2}$.

It is clear that in the special case $\gamma=1$ and $\alpha_{n\nu}=A_{n-\nu}^{(\alpha-1)}$ $(\alpha>0)$ this theorem includes the Theorem A of SUNOUCHI; in fact,

$$\left\{\sum_{\nu=1}^{n}\nu^{p-1}(A_{n-\nu}^{(\alpha-1)})^p\right\}^{1/p}\leqq K_1\{n^{p-1}n^{(\alpha-1)p+1}\}^{1/p}=K_1n^\alpha\leqq K_2A_n^{(\alpha)}$$

for any $\alpha>0$ if p is near enough to 1.

It is easy to verify that in the cases

(3.3)
$$\alpha_{nk} = k^{\beta},\quad \beta > -1;\quad \alpha_{nk} = \frac{1}{k};\quad \alpha_{nk} = \frac{1}{k\log(k+2)};$$
$$\alpha_{nk} = \frac{1}{k\log(k+2)\log\log(k+4)}$$

and in those cases, which are similar to the above ones, the condition (3.1) is satisfied for any $p>1$, consequently the statement (3.2) holds for any $k>0$ in the cases mentioned above.

It follows easily from this theorem:

Theorem 2. *Let $k>0$. If there exists a $p>1$ such that the conditions* (3.1) *holds and if*

(3.4)
$$\sum_{n=4}^{\infty} c_n^2 \log\log^2 n < \infty,$$

then we have

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_{\nu}^{\gamma-1}(\{\mu_i\}; x) - f(x)|^k = 0$$

almost everywhere in (a, b) for any $\gamma > \frac{1}{2}$ and for any increasing sequence $\{\mu_i\}$; here we have set

$$\sigma_n^{\beta}(\{\mu_i\}; x) = \frac{1}{A_n^{(\beta)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\beta-1)} s_{\mu_\nu}(x).$$

Theorem 2 includes evidently the Theorem B of SUNOUCHI in the special case $\gamma=1$ and $\alpha_{n\nu}=A_{n-\nu}^{(\alpha-1)}$ $(\alpha>0)$.

Theorem 3. *Under the hypothesis of Theorem 2 we have*

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |s_{l_\nu}(x) - f(x)|^k = 0$$

almost everywhere in (a, b) for any sequence $\{l_\nu\}$ of distinct non-negative integers.

In particular, we have as

Corollary 1. *If the condition* (3.4) *is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_{l_\nu}(x) - f(x)|^k = 0$$

holds for any $\alpha>0$ and $k>0$, almost everywhere in (a, b) for any sequence $\{l_\nu\}$ of distinct non-negative integers.

It is easy to see that this corollary generalizes the Theorems B and C.

Finally we prove the following

Theorem 4. *Let $\{d_n\}$ be a given real sequence with $\sum d_n^2 < \infty$ and*

(3.5)
$$n d_n^2 \geqq (n+1) d_{n+1}^2 \qquad (n=1, 2, \ldots),$$

further let $\gamma > \frac{1}{2}$ and $k > 0$. If there exists a $p > 1$ such that the conditions (3. 1) *hold, and if the series* (1. 1) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b) *and, moreover,*

$$c_n^2 = O(d_n^2), \tag{3.6}$$

then

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(x) - f(x)|^k = 0$$

almost everywhere in (a, b) *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

This theorem includes the Theorem D in the special case $\alpha_{n\nu}=1$ and $k=2$, because the conditions (3. 1) are satisfied in the cases of (3. 3) for any $p>1$, as we have seen it.

It seems worth while to observe also the following

Corollary 2. *Let* $\{d_n\}$ *be a given real sequence satisfying the conditions* $\sum d_n^2 < \infty$ *and* (3. 5). *If the series* (1. 1) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b) *and* (3. 6) *is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |\sigma_{l_\nu}^{\gamma-1}(x) - f(x)|^k = 0$$

holds for any $\alpha > 0$, $k > 0$, *and* $\gamma > \frac{1}{2}$, *almost everywhere in* (a, b), *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

The method of proof of these theorems is that of SUNOUCHI [3] and of the author [2].

In the sequel, we use $K, K_1, K_2, \ldots$ to denote positive constants, not necessarily the same on any two occurences.

4. The following lemmas will be required for the proofs of the theorems.

Lemma 1. *Let* $\{\psi_k(x)\}$ $(k=1, \ldots, N)$ *be an orthogonal system in* (a, b) *and let*

$$a_k^2 = \int_a^b \psi_k^2(x)\,dx \qquad (k=1, 2, \ldots, N).$$

Then there exists a function $\delta(x)$ *such that*

$$|\psi_1(x) + \ldots + \psi_l(x)| \leqq \delta(x) \qquad (l=1, 2, \ldots, N)$$

in (a, b) *and*

$$\int_a^b \delta^2(x)\,dx \leqq K_1 \log^2 N \sum_{k=1}^{N} a_k^2.$$

This Lemma is well known (cf. KACZMARZ—STEINHAUS [1], p. 162).

Lemma 2. *If* $\sum_{n=0}^{\infty} c_n^2 < \infty$, *then*

$$\int_a^b \left\{ \sum_{n=1}^{\infty} n^{-1} |\sigma_n^{\alpha-1}(x) - \sigma_n^{\alpha}(x)|^2 \right\} dx \leqq K_2 \sum_{n=0}^{\infty} c_n^2 \qquad \left(\alpha > \frac{1}{2}\right).$$

This Lemma also is known (cf. [3], Lemma 1).

Lemma 3. *Let $k>0$ and $\sum c_n^2<\infty$. If there exists a $p>1$ such that the conditions* (3. 1) *are satisfied, then for $\gamma>\frac{1}{2}$ we have*

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \right)^{1/k} \right\}^2 dx \leqq K_3 \sum_{n=0}^{\infty} c_n^2.$$

Proof. Applying HÖLDER's inequality, we obtain by (3. 1)

$$\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \leqq$$

$$\text{(4. 1)} \qquad \leqq \frac{1}{A_n} \left\{ \sum_{\nu=1}^{n} \nu^{-1} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk} \right\}^{1/q} \left\{ \sum_{\nu=1}^{n} \nu^{p/q} \alpha_{n\nu}^p \right\}^{1/p} \leqq$$

$$\leqq K \left\{ \sum_{\nu=1}^{n} \nu^{-1} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk} \right\}^{1/q},$$

where $q=\dfrac{p}{p-1}$. Since $qk\geqq 2$, we have by Lemma 2 and (4. 1) that

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \right)^{1/k} \right\}^2 dx \leqq$$

$$\leqq K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk} \right)^{2/qk} dx \leqq$$

$$\leqq K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^2 \right) dx \leqq K_2 \sum_{n=0}^{\infty} c_n^2.$$

5. Proof of Theorem 1. Since, by the hypothesis, the series (1. 1) is $(C, 1)$-summable, so the means $\sigma_n^\beta(x)$ $(\beta>0)$ converge to the function $f(x)$ almost everywhere in (a, b). From this fact it follows that in the following inequality

$$\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-f(x)|^k \leqq$$

$$\text{(5. 1)} \qquad \leqq \frac{K_1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k + \frac{K_1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^\gamma(x)-f(x)|^k$$

the second sum tends to 0 almost everywhere in (a, b).

We shall show that the first sum tends to zero, too. To this effect, we choose N, for given $\varepsilon>0$, so that

$$\text{(5. 2)} \qquad \sum_{n\geqq N/4}^{\infty} c_n^2 < \varepsilon^3,$$

and we consider the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \text{with} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N, \\ 0 & \text{for} \quad n > N, \end{cases} \tag{5. 3}$$

and

$$\sum_{n=1}^{\infty} b_n \varphi_n(x) \quad \text{with} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N, \\ c_n & \text{for} \quad n > N. \end{cases} \tag{5. 4}$$

Let us denote by $\sigma_\nu^\beta(a; x)$ and $\sigma_\nu^\beta(b; x)$, respectively, the ν-th Cesàro means of order β of the series (5. 3) and (5. 4). It is obvious that

$$\sigma_\nu^\beta(x) = \sigma_\nu^\beta(a; x) + \sigma_\nu^\beta(b; x) \qquad (\nu = 1, 2, \ldots). \tag{5. 5}$$

For the series (5. 3), the means

$$\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(a; x) - \sigma_\nu^\gamma(a; x)|^k$$

converge clearly to zero almost everywhere. As to the series (5. 4), we obtain, using the Lemma 3 and (5. 2),

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k \right)^{1/k} \right\}^2 \leqq K_1 \varepsilon^3.$$

Hence

$$\text{meas} \left\{ x \,\middle|\, \limsup \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K_1 \varepsilon.$$

That is, the means

$$\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k$$

also converge to zero almost everywhere.

The statement (3. 2) follows from the above results by virtue of (5. 5).

This completes the proof of Theorem 1.

Proof of Theorem 2. We set

$$\gamma_n^2 = \sum_{k=\mu_{n-1}+1}^{\mu_n} c_k^2$$

and

$$\Phi_n(x) = \begin{cases} \dfrac{1}{\gamma_n} \displaystyle\sum_{k=\mu_{n-1}+1}^{\mu_n} c_k \varphi_k(x) & \text{for} \quad \gamma_n \neq 0, \\ \dfrac{1}{\sqrt{\mu_n - \mu_{n-1}}} \displaystyle\sum_{k=\mu_{n-1}+1}^{\mu_n} \varphi_k(x) & \text{for} \quad \gamma_n = 0. \end{cases}$$

By (3. 4),

$$\sum_{n=4}^{\infty} \gamma_n^2 \log\log^2 n = \sum_{n=4}^{\infty} \log\log^2 n \sum_{k=\mu_{n-1}+1}^{\mu_n} c_k^2 < \infty.$$

Hence, and from a well known theorem of KACZMARZ and MENSHOV, it follows hat the series

$$\sum_{n=1}^{\infty} \gamma_n \Phi_n(x)$$

is $(C, 1)$-summable to $f(x)$ almost everywhere in (a, b). Applying the Theorem 1 to the above series, we obtain the statement of Theorem 2.

Proof of Theorem 3. Under the condition (3. 4) the sequence $\{s_{2^m}(x)\}$ converges to $f(x)$ almost everywhere in (a, b). We write

$$C_m^2 = \sum_{n=2^m+1}^{2^{m+1}} c_n^2.$$

Let $m(\geqq 2)$ be any natural number, for which $C_m \neq 0$. Set $\mu_0(m) = 2^m$ and let $\mu_i(m)$ $(1 \leqq i \leqq N_m)$ be the smallest natural number for which

$$\sum_{n=\mu_{i-1}(m)+1}^{\mu_i(m)} c_n^2 \geqq \frac{C_m^2}{m} \quad \text{and} \quad \mu_i(m) \leqq 2^{m+1}$$

are valid. It is clear that $N_m \leqq m$. If $C_m = 0$, we write $\mu_0(m) = 2^m$ and $\mu_1(m) = 2^{m+1}$. Let us apply Lemma 1 to the functions

$$\psi_i^{(m)}(x) = s_{\mu_i(m)}(x) - s_{\mu_{i-1}(m)}(x) \qquad (1 \leqq i \leqq N_m).$$

Thus there exists a function $\delta_m(x)$ such that

$$(5.6) \qquad |s_{\mu_i(m)}(x) - s_{2^m}(x)| = \left| \sum_{j=1}^{i} \psi_j^{(m)}(x) \right| \leqq \delta_m(x) \qquad (1 \leqq i \leqq N_m)$$

in (a, b) and

$$\int_a^b \delta_m^2(x)\,dx \leqq K_1 \log^2 m \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \leqq K_2 \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \log\log^2 n.$$

Then, by (3. 4),

$$\sum_{m=2}^{\infty} \int_a^b \delta_m^2(x)\,dx > \infty$$

hence the series

$$\sum_{m=2}^{\infty} \delta_m^2(x)$$

converges almost everywhere. This gives by (5. 6) that

$$s_{\mu_i(m)}(x) - s_{2^m}(x) \to 0$$

for $m \to \infty$, almost everywhere in (a, b). Hence also $s_{\mu_i(m)}(x)$ $(m \to \infty)$ converges to the function $f(x)$ almost everywhere in (a, b).

Let us now define the following sequence of indices $\{\mu_\nu\}$: if $\mu_i(m) \leqq l_\nu < \mu_{i+1}(m)$ then set $\mu_\nu = \mu_i(m)$, and if $\mu_{N_m}(m) \leqq l_\nu < \mu_0(m+1)$ then $\mu_\nu = \mu_{N_m}(m)$.

It is easy to see that

$$\text{(5.7)}\qquad \frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-f(x)|^k \leqq \leqq \frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k+\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{\mu_\nu}(x)-f(x)|^k.$$

Since $s_{\mu_\nu}(x)\to f(x)$ $(\nu\to\infty)$, the second sum tends to 0 almost everywhere in (a, b).

From this point on, the proof runs similarly to the proof of the Theorem 1. Let us define N, $\{a_n\}$ and $\{b_n\}$ in the same way as under (5. 2), (5. 3) and (5. 4). Let us denote by $s_n(a; x)$ and $s_n(b; x)$, respectively, the n-th partial sums of series (5. 3) and (5. 4). It is evident that

$$\text{(5.8)}\qquad s_n(x)=s_n(a;x)+s_n(b;x)\qquad (n=1, 2, \ldots).$$

We can see easily that

$$\text{(5.9)}\qquad \frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l\nu}(a;x)-s_{\mu_\nu}(a;x)|^k\to 0$$

almost everywhere in (a, b). An analogous statement for the series (5. 4), can be obtained by the following easy computation. Using HÖLDER's inequality and (3. 1), we obtain

$$\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k \leqq \leqq \frac{1}{A_n}\left\{\sum_{\nu=1}^{n}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right\}^{1/q}\left\{\sum_{\nu=1}^{n}\nu^{p/q}\alpha_{n\nu}^{p}\right\}^{1/p}\leqq \leqq K\left\{\sum_{\nu=1}^{n}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right\}^{1/q}.$$

Since $qk\geqq 2$, we have

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\right)^{1/k}\right\}^2 dx\leqq$$

$$\text{(5.10)}\qquad \leqq K_1\int_a^b\left(\sum_{\nu=1}^{\infty}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right)^{2/qk}dx\leqq$$

$$\leqq K_1\int_a^b\left(\sum_{\nu=1}^{\infty}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\right)dx.$$

An easy computation shows that *)

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b |s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\,dx = \sum_{\nu=1}^{\infty}\frac{1}{\nu}\sum_{k=\mu_\nu+1}^{l_\nu} b_k^2 =$$

$$= \sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_\nu<2^{m+1}}^{(\nu)}\frac{1}{\nu}\sum_{k=\mu_\nu+1}^{l_\nu} b_k^2 \leqq \sum_{m=[\log N]}^{\infty}\left(\sum_{2^m\leqq\mu_\nu<2^{m+1}}^{(\nu)}\frac{1}{\nu}\right)\frac{C_m^2}{m} \leqq$$

$$\leqq \sum_{m=[\log N]}^{\infty}\left(\sum_{\nu=1}^{2^m}\frac{1}{\nu}\right)\frac{C_m^2}{m} \leqq K_2\sum_{k\geqq N/2}^{\infty} c_k^2.$$

From this and (5. 10) it follows

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\right)^{1/k}\right\}^2 dx \leqq K_3\sum_{k\geqq N/2}^{\infty} c_k^2.$$

Hence

$$\text{meas}\left\{x\,\middle|\,\limsup\left(\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\right)^{1/k}>\varepsilon\right\}\leqq K_4\varepsilon.$$

From this we obtain that the means

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k$$

converge to zero almost everywhere in (a, b).

Hence and from (5. 9) by (5. 8) we get that

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k\to 0 \tag{5.11}$$

almost everywhere.

Finally, from (5. 7) and (5. 11) we obtain the statement of Theorem 3.

Proof of Theorem 4. By the hypothesis of the theorem, the series (1. 1) is $(C, 1)$-summable to $f(x)$, thus

$$\lim_{n\to\infty}\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\beta}(x)-f(x)|^k=0$$

almost everywhere in (a, b) for any $\beta>0$. Due to this fact, it suffices to prove the statement for the case $\frac{1}{2}<\gamma\leqq 1$. Since

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma-1}(x)-f(x)|^k\leqq$$

$$\leqq\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma-1}(x)-\sigma_{l_\nu}^{\gamma}(x)|^k+\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma}(x)-f(x)|^k,$$

we only have to show that the first sum tends to zero.

*) $\Sigma^{(\nu)}$ denotes that the sum is taken for ν. We use the logarithm with basis 2.

For any positive ε, we choose N so that

$$\sum_{n=N}^{\infty} d_n^2 < \varepsilon^3. \tag{5.12}$$

We define further $\{a_n\}$ and $\{b_n\}$ in the same way as under (5. 3) and (5. 4). Let $\sigma_\nu^\beta(a; x)$ and $\sigma_\nu^\beta(b; x)$ have the same meaning as in the proof of Theorem 1.

It is easy to see that

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(a; x) - \sigma_\nu^\gamma(a; x)|^k = 0$$

almost everywhere in (a, b). The analogous statement for the series (5. 4) is the basis of the proof of this theorem. After a computation analogous to the proof of Lemma 3, we get

$$\begin{aligned} &\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^\gamma(b; x)|^k \right)^{1/k} \right\}^2 dx \leqq \\ &\leqq K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^\gamma(b; x)|^2 \right) dx. \end{aligned} \tag{5.13}$$

An easy computation shows that

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m} \int_a^b |\sigma_{l_m}^{\gamma-1}(b; x) - \sigma_{l_m}^\gamma(b; x)|^2 \, dx \leqq \\ &\leqq K_1 \sum_{m=1}^{\infty} \frac{1}{m(A_{l_m}^{(\gamma)})^2} \sum_{k=1}^{l_m} (A_{l_m-k}^{(\gamma-1)})^2 k^2 b_k^2 \leqq K_2 \sum_{m=1}^{\infty} \frac{1}{m l_m^{2\gamma}} \sum_{k=1}^{l_m} (l_m - k + 1)^{2\gamma-2} k^2 b_k^2. \end{aligned} \tag{5.14}$$

Let us denote by m_i the ith natural number, for which $m_i \leqq l_{m_i}$, and by μ_n the nth natural number, for which $\mu_n > l_{\mu_n}$. Then we have

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m l_m^{2\gamma}} \sum_{k=1}^{l_m} (l_m - k + 1)^{2\gamma-2} k^2 b_k^2 = \\ &= \sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=1}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k^2 b_k^2 + \sum_{n=1}^{\infty} \frac{1}{\mu_n l_{\mu_n}^{2\gamma}} \sum_{k=1}^{l_{\mu_n}} (l_{\mu_n} - k + 1)^{2\gamma-2} k^2 b_k^2. \end{aligned} \tag{5.15}$$

Since $l_{m_i} \geqq m_i$, the first sum in (5. 15) is less than

$$K_3 \sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \left(\sum_{k=1}^{m_i-1} + \sum_{k=m_i}^{l_{m_i}} \right) (l_{m_i} - k + 1)^{2\gamma-2} k^2 b_k^2. \tag{5.16}$$

By virtue of (3. 5) and (3. 6), we have for $m_{i_0} \leqq N < m_{i_0+1}$

$$\begin{aligned} &\sum_{i=i_0}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=\max(m_i, N)}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k^2 d_k^2 \leqq \\ &\leqq \sum_{i=i_0}^{\infty} \frac{d_{\max(m_i, N)}^2}{l_{m_i}^{2\gamma}} \sum_{k=m_i}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k. \end{aligned} \tag{5.17}$$

Since $\gamma > \frac{1}{2}$ and $l_{m_i} \geqq m_i$, it holds

$$\sum_{k=m_i}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma - 2} k \leqq \sum_{p=1}^{l_{m_i} - m_i + 1} p^{2\gamma - 2} (l_{m_i} - p + 1) \leqq K_4 l_{m_i}^{2\gamma}.$$

Hence and from (5. 17) it follows

$$\sum_{i=i_0}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=\max(m_i N)}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma - 2} k^2 d_k^2 \leqq \tag{5.18}$$

$$\leqq K_5 \left(d_N^2 + \sum_{i=i_0+1}^{\infty} d_{m_i}^2 \right) \leqq K_5 \sum_{j=N}^{\infty} d_j^2.$$

Let i_k be the least natural number for which $m_{i_k} > k$. Since $\frac{1}{2} < \gamma \leqq 1$ and $l_{m_i} \geqq m_i$ $(i = 1, 2, \ldots)$, it follows

$$\sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=1}^{m_i - 1} (l_{m_i} - k + 1)^{2\gamma - 2} k^2 b_k^2 \leqq \sum_{k=N+1}^{\infty} k^2 d_k^2 \sum_{i=i_k}^{\infty} \frac{(l_{m_i} - k + 1)^{2\gamma - 2}}{m_i l_{m_i}^{2\gamma}} \leqq \tag{5.19}$$

$$\leqq \sum_{k=N}^{\infty} k d_k^2 \sum_{i=i_k}^{\infty} \frac{(m_i - k + 1)^{2\gamma - 2}}{m_i^{2\gamma}} \leqq \sum_{k=N}^{\infty} k d_k^2 \sum_{l=k}^{\infty} \frac{(l - k + 1)^{2\gamma - 2}}{l^{2\gamma}} \leqq K_6 \sum_{k=N}^{\infty} d_k^2.$$

We can estimate the second sum under (5. 15) more easily than the first one. In fact, considering that $\mu_n > l_{\mu_n}$ $(n = 1, 2, \ldots)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n l_{\mu_n}^{2\gamma}} \sum_{k=1}^{l_{\mu_n}} (l_{\mu_n} - k + 1)^{2\gamma - 2} k^2 b_k^2 \leqq K_1 \sum_{k=N}^{\infty} k^2 d_k^2 \sum_{l_{\mu_n} \geqq k}^{(n)} \frac{(l_{\mu_n} - k + 1)^{2\gamma - 2}}{\mu_n l_{\mu_n}^{2\gamma}} \leqq$$

$$\leqq K_1 \sum_{k=N}^{\infty} k d_k^2 \sum_{l=k}^{\infty} \frac{(l - k + 1)^{2\gamma - 2}}{l^{2\gamma}} \leqq K_2 \sum_{k=N}^{\infty} d_k^2.$$

Hence and from (5. 13)—(5. 19), considering (5. 12), we obtain that

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k \right)^{1/k} \right\}^2 dx \leqq K_3 \sum_{k=N}^{\infty} d_k^2 < \varepsilon^3. \tag{5.20}$$

The proof runs similarly to the proof of Theorem 3. From (5. 20) it follows that

$$\text{meas} \left\{ x \,\middle|\, \limsup \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K_1 \varepsilon,$$

i. e.

$$\lim_{n \to \infty} \frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k = 0$$

almost everywhere in (a, b).

This completes the proof of Theorem 4.

Literature

[1] S. KACZMARZ—H. STEINHAUS, *Theorie der Orthogonalreihen* (Warszawa—Lwów, 1935).
[2] L. LEINDLER, Über die starke Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **23** (1962), 82—89.
[3] G. SUNOUCHI, On the strong summability of orthogonal series, *Acta Sci. Math.*, **27** (1966), 71—76.
[4] K. TANDORI, Über die orthogonalen Funktionen. VI (Eine genaue Bedingung für die starke Summation), *Acta Sci. Math.*, **20** (1959), 14—18.

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