Ergodic type theorems in von Neumann algebras

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Introduction

Let $\mathbf{A}$ be a von Neumann algebra[1]) in a complex Hilbert space $\mathfrak{H}$, and let $\mathscr{G}$ be a group of automorphisms of $\mathbf{A}$ [2]). Denote by $\mathbf{A}^{\mathscr{G}}$ the set of all elements of $\mathbf{A}$ which are invariant with respect to each element of $\mathscr{G}$. Taking into account the algebraic and topological properties of the elements of $\mathscr{G}$ ([13], chap. I, § 4, Th. 2, Cor. 1), one can see easily that $\mathbf{A}^{\mathscr{G}}$ is a von Neumann subalgebra of $\mathbf{A}$. For any $T \in \mathbf{A}$, let $\mathscr{K}_0(T, \mathscr{G})$ denote the smallest convex subset of $\mathbf{A}$ which contains the *orbit* of T *under* $\mathscr{G}$ [3]). Let $\mathscr{K}(T, \mathscr{G})$ be the weak closure of $\mathscr{K}_0(T, \mathscr{G})$ [4]). The investigations concerning the center-valued trace theory of von Neumann algebras and the results of some other works (for example [1], [2], [7]) naturally give the idea of seeking conditions on $\mathbf{A}$ and $\mathscr{G}$ under which the set $\mathscr{K}(T, \mathscr{G})$ meets $\mathbf{A}^{\mathscr{G}}$ for every $T \in \mathbf{A}$.

The purpose of this paper is to give a sufficient condition in order that $\mathscr{K}(T, \mathscr{G}) \cap \mathbf{A}^{\mathscr{G}}$ consist of exactly one element for every $T \in \mathbf{A}$ (Theorem 1.) This is the subject of § 2. The next § 3 is devoted to establishing under this condition a mapping of $\mathbf{A}$ onto $\mathbf{A}^{\mathscr{G}}$ which reminds us, from many points of view, of the Dixmier trace $\natural$ of a finite von Neumann algebra (Theorem 2). In § 4, some simple consequences of the above results are given. § 1 contains preliminary results and examples.

The main results of this paper were announced in [5], with the proof of Theorem 1 in a less detailed form.

§ 1

First of all let us set down some notations.

If $\mathbf{A}$ is a von Neumann algebra and $\mathscr{G}$ is a group of automorphisms of $\mathbf{A}$, denote by $\mathscr{R}(\mathbf{A}, \mathscr{G})$ the set of all ultra-weakly continuous linear forms on $\mathbf{A}$ which

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[1]) For the theory of von Neumann algabras, cf. [3]. The terminology of [3] will be freely used in the following.

[2]) By an automorphism of a von Neumann algebra, we always mean a $*$-automorphism.

[3]) By the orbit of T under $\mathscr{G}$ we mean the set of the elements $\{\theta(T)\}_{\theta \in \mathscr{G}}$.

[4]) For a given pair $(\mathbf{A}, \mathscr{G})$, the notations $\mathscr{K}_0(T, \mathscr{G})$, $\mathscr{K}(T, \mathscr{G})$ $(T \in \mathbf{A})$ will be permanently used by us, without explaining again what they mean.

are *invariant with respect to* $\mathcal{G}$ (that is if $\sigma\in\mathcal{R}(\mathbf{A},\mathcal{G})$ then for every $T\in\mathbf{A}$ and $\theta\in\mathcal{G}$ we have $\sigma(\theta(T))=\sigma(T)$). Let $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ denote the set of all positive elements of $\mathcal{R}(\mathbf{A},\mathcal{G})$. For any element σ of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$, E_σ will denote the *support* of σ ([3], chap. I, § 4, Def. 3). It is easy to see that $E_\sigma\in\mathbf{A}^{\mathcal{G}}$. The group of all inner automorphisms of $\mathbf{A}$ will be denoted by $\mathcal{I}(\mathbf{A})$.

With these notations we have the following

Proposition 1. *Let* $\mathbf{A}$ *be a von Neumann algebra in a complex Hilbert space* $\mathfrak{H}$, *and let* $\mathcal{G}$ *be a group of automorphisms of* $\mathbf{A}$. *The following four conditions are equivalent:*

(i) *For every* $T\in\mathbf{A}^+$ [5]), $T\neq 0$ *there exists an element* σ *of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *such that* $\sigma(T)\neq 0$;

(ii) *For every* $T\in(\mathbf{A}^{\mathcal{G}})^+$, $T\neq 0$ *there exists an element* σ *of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *with* $\sigma(T)\neq 0$;

(iii) *There exists a family* $\{\sigma_\iota\}_{\iota\in I}$ *of elements of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *such that* $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ *for* $\iota\neq\varkappa$ *and* $\sum_{\iota\in I}E_{\sigma_\iota}=I_{\mathfrak{H}}$. [6])

(iv) $\sup_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})} E_\sigma=I_{\mathfrak{H}}$.

Proof. (i)⇒(ii) is evident.

(ii)⇒(iii). In fact, let $\{\sigma_\iota\}_{\iota\in I}$ be a maximal family of elements of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ such that $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ for $\iota\neq\varkappa$. Such a family exists by the Zorn's lemma. Set $E=\sum_{\iota\in I}E_{\sigma_\iota}$, and prove that $E=I_{\mathfrak{H}}$. To do this, suppose the contrary that is that $E\neq I_{\mathfrak{H}}$. Put $F=I_{\mathfrak{H}}-E$. Since $F\in(\mathbf{A}^{\mathcal{G}})^+$, $F\neq 0$, in virtue of (ii), there exists an element σ of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ such that $\sigma(F)\neq 0$. Set $\sigma'(T)=\sigma(FTF)$ for every $T\in\mathbf{A}$. As $F\in\mathbf{A}^{\mathcal{G}}$, we obtain that $\sigma'\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. Furthermore, we have $\sigma'\neq 0$ and $\sigma'(E)=0$. This means that $E_{\sigma'}\neq 0$ and $E_{\sigma'}\leqq F$, and this contradicts the maximality of the family $\{\sigma_\iota\}_{\iota\in I}$.

(iii)⇒(iv) is evident.

(iv)⇒(i). Suppose that (i) is not true. Then there exists an element $T\in\mathbf{A}^+$, $T\neq 0$ such that $\sigma(T)=0$ for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. This means that $E_\sigma TE_\sigma=0$ for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. Thus for every $x\in\mathfrak{H}$ we get $\|T^{\frac{1}{2}}E_\sigma x\|=0$, i.e. $T^{\frac{1}{2}}E_\sigma=0$. As, by (iv), $\sup_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})}E_\sigma=I_{\mathfrak{H}}$, we obtain that $T^{\frac{1}{2}}=0$, that is $T=0$ which is impossible, and this completes the proof of Proposition 1.

Definition 1. Let $\mathbf{A}$ be a von Neumann algebra and let $\mathcal{G}$ be a group of automorphisms of $\mathbf{A}$. $\mathbf{A}$ is said to be *finite with respect to* $\mathcal{G}$ (or $\mathcal{G}$*-finite*) if $\mathbf{A}$ and $\mathcal{G}$ satisfy any of the equivalent conditions of Proposition 1.

Remarks. 1. To say that $\mathbf{A}$ is $\mathcal{I}(\mathbf{A})$-finite is equivalent to say that $\mathbf{A}$ is finite in the usual sense of the global theory of the von Neumann algebras ([3], chap. I, § 6, Def. 5).

2. If $\mathbf{A}$ is $\mathcal{G}$-finite then $\mathbf{A}$ is finite with respect to any subgroup of $\mathcal{G}$.

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5) For a von Neumann algebra $\mathbf{A}$, $\mathbf{A}^+$ denotes the set of all non-negative self-adjoint elements of $\mathbf{A}$.

6) $I_{\mathfrak{H}}$ denotes the identity operator of the Hilbert space $\mathfrak{H}$.

Now let us give *examples* for pairs $(\mathbf{A}, \mathcal{G})$ such that $\mathbf{A}$ is $\mathcal{G}$-finite.

1. $\mathbf{A}$ *is a finite von Neumann algebra and* $\mathcal{G}$ *is an arbitrary subgroup of* $\mathcal{I}(\mathbf{A})$.

2. $\mathbf{A}$ *is a finite factor and* $\mathcal{G}$ *is an arbitrary group of automorphisms of* $\mathbf{A}$. In fact, if $\mathrm{Tr}(\cdot)$ is the canonical trace of $\mathbf{A}$ ([3], chap. III, no. 4) and θ is an arbitrary element of $\mathcal{G}$ then $\varphi(T)=\mathrm{Tr}\,(\theta(T))$ $(T\in\mathbf{A})$ is also a normalized trace[7]) on $\mathbf{A}$. Therefore, for every $T\in\mathbf{A}$ we have $\mathrm{Tr}\,(T)=\varphi(T)=\mathrm{Tr}\,(\theta(T))$ ([3], chap. I. § 6, Th. 3, Cor.), and this means that $\mathrm{Tr}(\cdot)\in\mathcal{R}^+(\mathbf{A}, \mathcal{G})$. Since $\mathrm{Tr}(\cdot)$ is a strictly positive linear form on $\mathbf{A}$, we obtain that $\mathbf{A}$ is $\mathcal{G}$-finite.

3. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be von Neumann algebras in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let $\mathcal{G}_i$ be a group of automorphisms of $\mathbf{A}_i$ for every $i=1,2$. Put $\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$ and $\mathbf{A}=\mathbf{A}_1\otimes\mathbf{A}_2$. If $\theta_1\in\mathcal{G}_1$ and $\theta_2\in\mathcal{G}_2$, there exists a uniquely defined automorphism θ of $\mathbf{A}$ such that $\theta(T_1\otimes T_2)=\theta_1(T_1)\otimes\theta_2(T_2)$ for every $T_1\in\mathbf{A}_1$ and $T_2\in\mathbf{A}_2$ ([3], chap. I, § 4. Prop. 2). Denote by $\mathcal{G}_1\otimes\mathcal{G}_2$ the set of all θ obtained from all possible pairs $\{\theta_1\in\mathcal{G}_1, \theta_2\in\mathcal{G}_2\}$ in this way. Under the usual multiplication, $\mathcal{G}=\mathcal{G}_1\otimes\mathcal{G}_2$ is a group of automorphisms of $\mathbf{A}$.

Proposition 2. *If* $\mathbf{A}_1$ *is* $\mathcal{G}_1$*-finite and* $\mathbf{A}_2$ *is* $\mathcal{G}_2$*-finite then* $\mathbf{A}$ *is* $\mathcal{G}$*-finite.*

Proof. In virtue of Definition 1, it is enough to show that $\sup\limits_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})} E_\sigma=I_{\mathfrak{H}}$. To do this, consider an arbitrary element $\sigma_i\in\mathcal{R}^+(\mathbf{A}_i, \mathcal{G}_i)$ $(i=1, 2)$. It is known ([3], chap. I, § 4, Th. 1) that for each $i=1, 2$, there exists a sequence $\{x_k^{(i)}\}_{k=1}^\infty$ of elements of $\mathfrak{H}_i$ with $\sum\limits_{k=1}^\infty \|x_k^{(i)}\|^2<+\infty$ such that for every $T_i\in\mathbf{A}_i$ we have

$$\sigma_i(T_i)=\sum_{k=1}^\infty (T_i x_k^{(i)}\,|\,x_k^{(i)}).$$

Now for every $T\in\mathbf{A}$, put

$$\sigma(T)=\sum_{k=1}^\infty\sum_{l=1}^\infty (T[x_k^{(1)}\otimes x_l^{(2)}]\,|\,x_k^{(1)}\otimes x_l^{(2)}).$$

It is easy to see that $\sigma(T_1\otimes T_2)=\sigma_1(T_1)\sigma_2(T_2)$ for every $T_1\in\mathbf{A}_1, T_2\in\mathbf{A}_2$. By linearity and continuity, from this we can conclude that $\sigma\in\mathcal{R}^+(\mathbf{A}, \mathcal{G})$. Furthermore,

$$E_{\sigma_1}\mathfrak{H}_1=\mathfrak{X}^{\mathbf{A}_1'}_{\{x_k^{(1)}\}_{k=1}^\infty},\qquad E_{\sigma_2}\mathfrak{H}_2=\mathfrak{X}^{\mathbf{A}_2'}_{\{x_k^{(2)}\}_{k=1}^\infty}\quad\text{and}\quad E_\sigma\mathfrak{H}=\mathfrak{X}^{\mathbf{A}'}_{\{x_k^{(1)}\otimes x_l^{(2)}\}_{k,l=1}^\infty}\ {}^{8)}$$

([3], chap. I, § 4, no. 6).

On the other hand, we have $\mathbf{A}_1'\otimes\mathbf{A}_2'\subseteq\mathbf{A}'$. This implies that

$$E_{\sigma_1}\otimes E_{\sigma_2}\leqq E_\sigma. \tag{1.1}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are $\mathcal{G}_1$- and $\mathcal{G}_2$-finite, respectively, we have that

$$\sup_{\sigma_1\in\mathcal{R}^+(\mathbf{A}_1,\mathcal{G}_1),\ \sigma_2\in\mathcal{R}^+(\mathbf{A}_2,\mathcal{G}_2)} E_{\sigma_1}\otimes E_{\sigma_2}=I_{\mathfrak{H}}.$$

This together with (1.1) gives that $\sup\limits_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})} E_\sigma=I_{\mathfrak{H}}$, and so the proof of Proposition 2 is complete.[9])

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[7]) That is, $\varphi(I_{\mathfrak{H}})=1$.

[8]) For these notations, cf. [3], chap. I, § 1, no. 4.

[9]) For this reasoning, see [3], chap. I, § 4, Ex. 6.

Proposition 2 enables us to give examples for pairs $(\mathbf{A}, \mathcal{G})$ such that $\mathbf{A}$ is purely infinite ([3], chap. I, § 6, Def. 5), $\mathcal{G}$ is a non-trivial group of automorphisms[10]) of $\mathbf{A}$, and $\mathbf{A}$ is $\mathcal{G}$-finite. For instance, let $\mathbf{M}_1$ be a finite factor, and let $\mathcal{G}_1$ be an arbitrary but non-trivial group of automorphisms of $\mathbf{M}_1$. Let $\mathbf{M}_2$ be a purely infinite von Neumann algebra. Then $\mathbf{A}=\mathbf{M}_1\otimes\mathbf{M}_2$ is purely infinite ([6]). Put $\mathcal{G}=\mathcal{G}_1\otimes\mathcal{J}$, where $\mathcal{J}$ is the trivial group of automorphisms of $\mathbf{M}_2$. Then $\mathcal{G}$ is a non-trivial group of automorphisms of $\mathbf{A}$ and $\mathbf{A}$ is $\mathcal{G}$-finite (cf. Ex. 2 above and Prop. 2).

§ 2

Our main result can be stated as follows.

Theorem 1. *Let* $\mathbf{A}$ *be a von Neumann algebra and let* $\mathcal{G}$ *be a group of automorphisms of* $\mathbf{A}$. *Suppose that* $\mathbf{A}$ *is* $\mathcal{G}$*-finite. Then for every* $T\in\mathbf{A}$, $\mathcal{K}(T, \mathcal{G})\cap\mathbf{A}^{\mathcal{G}}$ *consists of exactly one element.*

A key-role in the proof of this theorem is played by the ergodic theorem of Alaoglu and Birkhoff ([4], Th. 1. 1. 3.). For convenience, we recall the reader just for a particular part of it we need.

Lemma 1. *Let* $\mathfrak{H}$ *be a complex Hilbert space, and let* $\mathcal{U}$ *be a group of unitary operators in* $\mathfrak{H}$. *For an arbitrary* $x\in\mathfrak{H}$, *denote by* $c(x, \mathcal{U})$ *the smallest convex subset of* $\mathfrak{H}$ *which contains the orbit of* x *under* $\mathcal{U}$. *Let* $\bar{c}(x, \mathcal{U})$ *be the closure of* $c(x, \mathcal{U})$ *in* $\mathfrak{H}$. *Then there exists a unique element* x_0 *in* $\bar{c}(x, \mathcal{U})$ *such that* $Ux_0=x_0$ *for every* $U\in\mathcal{U}$. *The mapping* $x\to x_0$ *is linear.*

Proof of Theorem 1. Let T be an arbitrary but fixed element of $\mathbf{A}$, and consider an arbitrary σ in $\mathcal{R}^+(\mathbf{A}, \mathcal{G})$. As σ is ultra-weakly continuous,

$$\mathfrak{m}_\sigma=\{S\in\mathbf{A}\colon \sigma(S^*S)=0\}$$

is an ultra-weakly closed left ideal of A. Consider the quotient vector space $\mathbf{A}/\mathfrak{m}_\sigma$, and let $S\to\eta_\sigma(S)$ denote the canonical mapping of $\mathbf{A}$ onto $\mathbf{A}/\mathfrak{m}_\sigma$. For every $R, S\in\mathbf{A}$, set

$$\langle\eta_\sigma(R)|\eta_\sigma(S)\rangle_\sigma=\sigma(S^*R). \tag{2. 1}$$

Then the vector space $\mathbf{A}/\mathfrak{m}_\sigma$ becomes a pre-Hilbert space with respect to the inner product (2. 1). Let $\mathfrak{H}_\sigma$ be the completion of $\mathbf{A}/\mathfrak{m}_\sigma$ in the norm defined by (2. 1).[11]) Now, let θ be an arbitrary element of $\mathcal{G}$. For any $\eta_\sigma(S)\in\mathbf{A}/\mathfrak{m}_\sigma$ $(S\in\mathbf{A})$, put

$$\overset{(\sigma)}{\theta_0}\,\eta_\sigma(S) = \eta_\sigma(\theta(S)). \tag{2. 2}$$

First of all we note that $\overset{(\sigma)}{\theta_0}$ is uniquely defined, that is its definition does not depend on the special choice of the representatives of the elements of $\mathbf{A}/\mathfrak{m}_\sigma$. Indeed,

[10]) That is $\mathcal{G}$ does not consists just of the identical automorphism of $\mathbf{A}$.

[11]) For this construction, see [3], chap. I, § 4, no. 1.

since σ is invariant with respect to θ, θ sends $\mathfrak{m}_\sigma$ onto itself. So, if S_1 and S_2 are two elements of **A** such that $\eta_\sigma(S_1)=\eta_\sigma(S_2)$ then $S_1-S_2\in\mathfrak{m}_\sigma$ and

$$\overset{(\sigma)}{\theta}_0\eta_\sigma(S_1)-\overset{(\sigma)}{\theta}_0\eta_\sigma(S_2)=\eta_\sigma(\theta(S_1))-\eta_\sigma(\theta(S_2))=\eta_\sigma(\theta(S_1-S_2))=0,$$

which means that $\overset{(\sigma)}{\theta}_0\,\eta_\sigma(S_1)=\overset{(\sigma)}{\theta}_0\,\eta_\sigma(S_2)$. It is clear that $\overset{(\sigma)}{\theta}_0$ is linear. Furthermore, $\overset{(\sigma)}{\theta}_0\,(\mathbf{A}/\mathfrak{m}_\sigma)\subseteq\mathbf{A}/\mathfrak{m}_\sigma$ by definition. Now, if $\eta_\sigma(S)$ is an arbitrary element of $\mathbf{A}/\mathfrak{m}_\sigma$, then $\overset{(\sigma)}{\theta}_0\,\eta_\sigma(\theta^{-1}(S))=\eta_\sigma(S)$ which means that $\overset{(\sigma)}{\theta}_0$ is surjective.

Consider now two arbitrary elements S_1 and S_2 of **A**. Then we have

$$\begin{aligned}\langle\overset{(\sigma)}{\theta}_0\eta_\sigma(S_1)\,|\,\overset{(\sigma)}{\theta}_0\eta_\sigma(S_2)\rangle_\sigma &= \sigma(\theta(S_2^*)\theta(S_1))=\sigma(\theta(S_2^*S_1))=\\ &=\sigma(S_2^*S_1)=\langle\eta_\sigma(S_1)\,|\,\eta_\sigma(S_2)\rangle_\sigma.\end{aligned}\tag{2.3}$$

Therefore, $\overset{(\sigma)}{\theta}_0$ can be uniquely extended to a unitary operator $\overset{(\sigma)}{\theta}$ of $\mathfrak{H}_\sigma$. Furthermore, it is not hard to prove that $[\overset{(\sigma)}{\theta}]^*=(\theta^{-1})^{(\sigma)}$, and that the family $\{\overset{(\sigma)}{\theta}\}_{\theta\in\mathcal{G}}$ is a group under the usual multiplication of unitary operators. Denote this group by $\overset{(\sigma)}{\mathcal{G}}$. Now, applying Lemma 1 to $\mathfrak{H}_\sigma$ and $\overset{(\sigma)}{\mathcal{G}}$, we obtain a unique point, say $\overset{(\sigma)}{x}$, in $\bar{c}(\eta_\sigma(T),\overset{(\sigma)}{\mathcal{G}})$ such that

$$\overset{(\sigma)}{\theta}\,\overset{(\sigma)}{x}=\overset{(\sigma)}{x}\tag{2.4}$$

for every $\overset{(\sigma)}{\theta}\in\overset{(\sigma)}{\mathcal{G}}$. We are going to prove that $\overset{(\sigma)}{x}\in\mathbf{A}/\mathfrak{m}_\sigma$. To do this, consider a sequence $\{x_n\}_{n=1}^\infty$ of elements of $c(\eta_\sigma(T),\overset{(\sigma)}{\mathcal{G}})$ with $\|x_n-\overset{(\sigma)}{x}\|_\sigma\to 0$ if $n\to\infty$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of elements of $\mathcal{K}_0(T,\mathcal{G})$ such that $\eta_\sigma(T_n)=x_n$ for every $n=1,2,\ldots$. Then we have

$$\sigma((T_m-T_n)^*(T_m-T_n))=\|\eta_\sigma(T_m)-\eta_\sigma(T_n)\|_\sigma^2=\|x_m-x_n\|_\sigma^2\to 0\tag{2.5}$$

for $m,n\to\infty$. As $\|T_m-T_n\|\leqq 2\|T\|$ [12]), in virtue of [3], chap. I, § 4, Prop. 4, we conclude from (2.5) that $(T_m-T_n)E_\sigma\to 0$ strongly for $m,n\to\infty$. Therefore, there exists a well-defined element S_1 of A such that

$$T_nE_\sigma\to S_1\tag{2.6}$$

strongly for $n\to\infty$. Now, as $\|T_nE_\sigma-S_1\|\leqq 2\|T\|$ $(n=1,2,\ldots)$, using again the proposition of [3] which has just been quoted, we obtain that

$$\|x_n-\eta_\sigma(S_1)\|_\sigma^2=\|\eta_\sigma(T_n)-\eta_\sigma(S_1)\|_\sigma^2=\sigma((T_n-S_1)^*(T_n-S_1))\to 0\tag{2.7}$$

for $m,n\to\infty$. So,

$$\overset{(\sigma)}{x}=\eta_\sigma(S_1)\quad\text{with}\quad S_1\in\mathbf{A},\tag{2.8}$$

[12]) $\|\ \|$ denotes the usual norm of bounded linear operators.

that is

$$\overset{(\sigma)}{x} \in \mathbf{A}/\mathfrak{m}_\sigma. \tag{2.9}$$

As the ultra-weak topology is compatible with the vector space structure of $\mathbf{A}$ and $\mathfrak{m}_\sigma$ is ultra-weakly closed, the set $\overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x})$ is ultra-weakly closed in $\mathbf{A}$. Set

$$\mathbf{A}^t_\sigma(T) = \overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x}) \cap \mathbf{A}_t \tag{2.10}$$

where $t = \|T\|$ and $\mathbf{A}_t = \{S \in \mathbf{A}: \|S\| \leqq t\}$. Then $\mathbf{A}^t_\sigma(T)$ is weakly closed as the weak topology coincides with the ultra-weak one on norm-bounded parts of $\mathbf{A}$. Furthermore, $\mathbf{A}^t_\sigma(T)$ is not empty as it contains at least S_1 constructed above (see (2. 8)). As a next step of our proof, let us construct the set $\mathbf{A}^t_\sigma(T)$ for every $\sigma \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$. Then, if $\sigma_1, \sigma_2 \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$, we have

$$\mathbf{A}^t_{\sigma_1+\sigma_2}(T) \subseteqq \mathbf{A}^t_{\sigma_i}(T) \qquad (i=1, 2). \tag{2.11}$$

Since $\sigma_1 + \sigma_2 \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$ and $\sigma_1 + \sigma_2 \geqq \sigma_i$ [13]) $(i=1, 2)$, to prove (2. 11) we have to show that if $\sigma', \sigma'' \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$ with $\sigma' \leqq \sigma''$ then $\mathbf{A}^t_{\sigma''}(T) \subseteqq \mathbf{A}^t_{\sigma'}(T)$. Well, suppose that we are given σ', σ'' from $\mathcal{R}^+(\mathbf{A}, \mathcal{G})$ with $\sigma' \leqq \sigma''$, and take an arbitrary element S of $\mathbf{A}^t_{\sigma''}(T)$. We have to prove that $S \in \mathbf{A}^t_{\sigma'}(T)$. First we note that $S \in \mathbf{A}^t_{\sigma''}(T)$ implies $\|S\| \leqq t$. So to show that $S \in \mathbf{A}^t_{\sigma'}(T)$, it suffices to prove that $\eta_{\sigma'}(S) = \overset{(\sigma')}{x}$ (where $\overset{(\sigma')}{x}$ plays the same role in the case of σ' as $\overset{(\sigma)}{x}$ did in the case of σ). Let $\{T_n\}_{n=1}^\infty$ be a sequence of elements of $\mathcal{K}_0(T, \mathcal{G})$ such that

$$\|\eta_{\sigma''}(T_n) - \overset{(\sigma'')}{x}\|_{\sigma''} \to 0 \qquad (n \to \infty).$$

By our assumption, $S \in \mathbf{A}^t_{\sigma''}(T)$ that is $\eta_{\sigma''}(S) = \overset{(\sigma'')}{x}$. Therefore, we have

$$\|\eta_{\sigma'}(T_n) - \eta_{\sigma'}(S)\|^2_{\sigma'} = \sigma'\big((T_n - S)^*(T_n - S)\big) \leqq$$

$$\leqq \sigma''\big((T_n - S)^*(T_n - S)\big) = \|\eta_{\sigma''}(T_n) - \eta_{\sigma''}(S)\|^2_{\sigma''} = \|\eta_{\sigma''}(T_n) - \overset{(\sigma'')}{x}\|^2_{\sigma''} \to 0$$

if $n \to \infty$. So we obtain that $\eta_{\sigma'}(S) \in \bar{c}(\eta_{\sigma'}(T), \overset{(\sigma')}{\mathcal{G}})$, and it remains to prove that $\eta_{\sigma'}(S)$ is invariant with respect to each element of $\overset{(\sigma')}{\mathcal{G}}$. Let $\overset{(\sigma')}{\theta} \in \overset{(\sigma')}{\mathcal{G}}$ be arbitrary. Then

$$\|\overset{(\sigma')}{\theta}\eta_{\sigma'}(S) - \eta_{\sigma'}(R)\|_{\sigma'} = \|\eta_{\sigma'}(\theta(S)) - \eta_{\sigma'}(S)\|_{\sigma'} \leqq$$

$$\leqq \|\eta_{\sigma''}(\theta(S)) - \eta_{\sigma''}(S)\|_{\sigma''} = \|\overset{(\sigma'')}{\theta}\,\overset{(\sigma'')}{x} - \overset{(\sigma'')}{x}\|_{\sigma''} = 0.$$

So $\overset{(\sigma')}{\theta}\eta_{\sigma'}(S) = \eta_{\sigma'}(S)$ for every $\overset{(\sigma')}{\theta} \in \overset{(\sigma')}{\mathcal{G}}$. Using the uniqueness of $\overset{(\sigma')}{x}$ in $\bar{c}(\eta_{\sigma'}(T), \overset{(\sigma')}{\mathcal{G}})$ we get that $\eta_{\sigma'}(S) = \overset{(\sigma')}{x}$, indeed. Hence (2. 11) is proved. In virtue of (2. 11), the

[13]) That means that $\sigma_1(T) + \sigma_2(T) \geqq \sigma_i(T)$ $(i=1, 2)$ for every $T \in \mathbf{A}^+$.

amily $\{\mathbf{A}_\sigma^t(T)\}_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})}$ is a filter basis on $\mathbf{A}_t$. It is known that $\mathbf{A}_t$ is weakly ompact ([3], chap. I, § 3, Th. 2). Thus, as each $\mathbf{A}_\sigma^t(T)$ is weakly closed, we obtain that

(2.12) $$\mathbf{A}^t(T)=\bigcap_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})}\mathbf{A}_\sigma^t(T)\neq\emptyset.$$

Now put

(2.13) $$\mathbf{A}_\sigma(T)=\overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x})$$

for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$. Then

(2.14) $$\mathbf{A}(T)=\bigcap_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})}\mathbf{A}_\sigma(T)$$

is not empty since $\mathbf{A}_\sigma^t(T)\subseteqq\mathbf{A}_\sigma(T)$ for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ and (2.12) holds. Now if $S_1\in\mathbf{A}(T)$ and $S_2\in\mathbf{A}(T)$, then for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ we obtain that

$$\eta_\sigma(S_1)=\eta_\sigma(S_2)=\overset{(\sigma)}{x},$$

hence $\sigma((S_1-S_2)^*(S_1-S_2))=0$. As $\mathbf{A}$ is supposed to be $\mathscr{G}$-finite, we get that $S_1=S_2$. This means that $\mathbf{A}(T)=\mathbf{A}^t(T)$, and it consists of exactly one element. Denote this unique element by $T^{\mathscr{G}}$. We are going to show that

(2.14) $$\mathscr{K}(T,\mathscr{G})\cap\mathbf{A}^{\mathscr{G}}=\{T^{\mathscr{G}}\},$$

where $\{T^{\mathscr{G}}\}$ denotes the set consisting of the element $T^{\mathscr{G}}$ alone. To do this, consider an arbitrary element θ of $\mathscr{G}$. For every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ we have

$$\sigma\big((\theta(T^{\mathscr{G}})-T^{\mathscr{G}})^*(\theta(T^{\mathscr{G}})-T^{\mathscr{G}})\big)=\|\eta_\sigma(\theta(T^{\mathscr{G}}))-\eta_\sigma(T^{\mathscr{G}})\|_\sigma^2=$$
$$=\|\overset{(\sigma)}{\theta}\overset{(\sigma)}{x}-\overset{(\sigma)}{x}\|_\sigma^2=0.$$

Hence $\theta(T^{\mathscr{G}})=T^{\mathscr{G}}$ which gives

(2.15) $$T^{\mathscr{G}}\in\mathbf{A}^{\mathscr{G}}.$$

Now let $x_1,\dots,x_n$ be an arbitrary finite family of elements of $\mathfrak{H}$. Then there exists an element σ_0 of $\mathscr{R}^+(\mathbf{A},\mathscr{G})$ such that $E_{\sigma_0}x_i=x_i$ for every $i=1,\dots,n$. In fact, consider a family $\{\sigma_\iota\}_{\iota\in I}$ of elements of $\mathscr{R}^+(\mathbf{A},\mathscr{G})$ with $\sigma_\iota(I_{\mathfrak{H}})=1$ $(\iota\in I)$, $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ for $\iota\neq\varkappa$, and $\sum_{\iota\in I}E_{\sigma_\iota}=I_{\mathfrak{H}}$. Then there exists a countable subfamily $\{\sigma_{\iota_n}\}_{n=1}^\infty$ of $\{\sigma_\iota\}_{\iota\in I}$ such that $\left(\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\right)x_i=x_i$ $(i=1,\dots,n)$. For every $T\in\mathbf{A}$ put

$$\sigma_0(T)=\sum_{n=1}^\infty\frac{1}{2^n}\sigma_{\iota_n}(T).$$

It is clear that $\sigma_0\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ ([3], chap. I, § 3, no. 3). Furthermore, if for a projection P of $\mathbf{A}$ we have $\sigma_0(P)=0$, then $\sigma_{\iota_n}(P)=0$ for every $n=1,2,\dots$. This means that $\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\leqq E_{\sigma_0}$. On the other hand,

$$\sigma_0\left(E_{\sigma_0}-\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\right)=\sum_{n=1}^\infty\frac{1}{2^n}[\sigma_{\iota_n}(E_{\sigma_0})-\sigma_{\iota_n}(E_{\sigma_{\iota_n}})]=$$
$$=\sum_{n=1}^\infty\frac{1}{2^n}[\sigma_{\iota_n}(E_{\sigma_{\iota_n}})-\sigma_{\iota_n}(E_{\sigma_{\iota_n}})]=0.$$

From this it follows that $E_{\sigma_0}-\sum_{n=1}^{\infty} E_{\sigma_{\iota_n}} \leqq I-E_{\sigma_0}$, which gives that $E_{\sigma_0}-\sum_{n=1}^{\infty} E_{\sigma_{\iota_n}}=0$. So $E_{\sigma_0}=\sum_{n=1}^{\infty} E_{\sigma_{\iota_n}}$, that is $E_{\sigma_0} x_i=x_i$ $(i=1,2,\ldots,n)$. Now let $\{T_m\}_{m=1}^{\infty}$ be a sequence of elements of $\mathcal{K}_0(T,\mathcal{G})$ such that $\|\eta_{\sigma_0}(T_m)-\eta_{\sigma_0}(T^{\mathcal{G}})\|_{\sigma_0}\to 0$ for $m\to\infty$. This implies that

$$(T_m-T^{\mathcal{G}})E_{\sigma_0}\to 0$$

strongly for $m\to\infty$ ([3], chap. I, § 4, Prop 4). Thus, for every $\varepsilon>0$ there exists an index $m_0=m_0(\varepsilon)$ such that

$$\|(T_{m_0}-T^{\mathcal{G}})E_{\sigma_0}x_i\|<\varepsilon \qquad (i=1,\ldots,n).$$

As $E_{\sigma_0}x_i=x_i$ $(i=1,\ldots,n)$, we get that

$$\|(T_{m_0}-T^{\mathcal{G}})x_i\|<\varepsilon \qquad (i=1,\ldots,n).$$

Hence, $T^{\mathcal{G}}\in\mathcal{K}(T,\mathcal{G})$, as the strong closure and the weak closure of $\mathcal{K}_0(T,\mathcal{G})$ coincide ([3], chap. I, § 3, Th. 1). Thus we have proved that

$$\{T^{\mathcal{G}}\}\subseteqq\mathcal{K}(T,\mathcal{G})\cap\mathbf{A}^{\mathcal{G}}. \tag{2.16}$$

Now let S be an arbitrary element of $\mathcal{K}(T,\mathcal{G})\cap A^{\mathcal{G}}$. Then using again [3], chap. I, § 4, Prop. 4, it is not hard to see that for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$ we have $\eta_\sigma(S)\in\bar{c}(\eta_\sigma(T),\overset{(\sigma)}{\mathcal{G}})$ and $\eta_\sigma(S)$ is invariant with respect to the elements of $\overset{(\sigma)}{\mathcal{G}}$. Therefore, we have $\eta_\sigma(S)=\overset{(\sigma)}{x}$ for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. Hence we obtain that $S\in A(T)=\{T^{\mathcal{G}}\}$, that is

$$\mathcal{K}(T,\mathcal{G})\cap\mathbf{A}^{\mathcal{G}}\subseteqq\{T^{\mathcal{G}}\}, \tag{2.17}$$

which implies, together with (2. 16), that

$$\{T^{\mathcal{G}}\}=\mathcal{K}(T,\mathcal{G})\cap\mathbf{A}^{\mathcal{G}}. \tag{2.18}$$

Since T was arbitrary in $\mathbf{A}$, Theorem 1 is completely proved.

§ 3

Now we are in the position to prove

Theorem 2. *Let* $\mathbf{A}$ *be a von Neumann algebra in a complex Hilbert space* $\mathfrak{H}$, *and let* $\mathcal{G}$ *be a group of automorphisms of* $\mathbf{A}$. *Suppose that* $\mathbf{A}$ *is* $\mathcal{G}$*-finite. Then the mapping* $T\to T^{\mathcal{G}}$[14]) *possesses the following properties:*

(i) *for every* $\sigma\in\mathcal{R}(\mathbf{A},\mathcal{G})$ *and* $T\in\mathbf{A}$ *we have* $\sigma(T)=\sigma(T^{\mathcal{G}})$;

(ii) $T\to T^{\mathcal{G}}$ *is linear and strictly positive;*[15])

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[14]) $T^{\mathcal{G}}$, as above, denotes the unique element of $\mathcal{K}(T,\mathcal{G})\cap\mathbf{A}^{\mathcal{G}}$ (cf. Th. 1).

[15]) In general, if $T\to\Phi(T)$ is a mapping of $\mathbf{A}$ into itself, Φ is said to be *positive* if $T\in\mathbf{A}^+$ implies $\Phi(T)\in\mathbf{A}^+$. Φ is *strictly positive*, if $T\in\mathbf{A}^+$, $T\neq O$ imply $\Phi(T)\geqq O$, $\Phi(T)\neq O$.

(iii) *if* $T\in\mathbf{A}$, $S\in\mathbf{A}^{\mathscr{G}}$ *we have* $(ST)^{\mathscr{G}}=ST^{\mathscr{G}}$ *and* $(TS)^{\mathscr{G}}=T^{\mathscr{G}}S$;
(iv) $T\to T^{\mathscr{G}}$ *is ultra-weakly and ultra-strongly continuous*;
(v) *for every* $T\in\mathbf{A}^{\mathscr{G}}$ *we have* $T=T^{\mathscr{G}}$;
(vi) $(\theta(T))^{\mathscr{G}}=T^{\mathscr{G}}$ *for every* $T\in\mathbf{A}$ *and* $\theta\in\mathscr{G}$.

Conversely, if we do not suppose that $\mathbf{A}$ *is* $\mathscr{G}$*-finite but we know that there exists an ultra-weakly continuous positive linear mapping* $T\to T'$ *of* $\mathbf{A}$ *onto* $\mathbf{A}^{\mathscr{G}}$ *such that*
a) $T=T'$ *for every* $T\in\mathbf{A}^{\mathscr{G}}$,
b) $(\theta(T))'=T$ *for every* $T\in\mathbf{A}$, $\theta\in\mathscr{G}$,
then $\mathbf{A}$ *is necessarily* $\mathscr{G}$*-finite and for every* $T\in\mathbf{A}$ *we have* $T'=T^{\mathscr{G}}$ (*cf.*[14])*.*

Proof. (i) It suffices to take into account the construction of $T^{\mathscr{G}}$ and to note that if $\sigma\in\mathscr{R}(\mathbf{A},\mathscr{G})$ then σ is weakly continuous on every norm-bounded part of $\mathbf{A}$, in particular on $\mathscr{K}(T,\mathscr{G})$.

(ii) Consider two arbitrary elements S and T of $\mathbf{A}$. Then we have $S^{\mathscr{G}}+T^{\mathscr{G}}\in\mathbf{A}^{\mathscr{G}}$. We are going to prove that $S^{\mathscr{G}}+T^{\mathscr{G}}$ belongs to $\mathscr{K}(S+T,\mathscr{G})$, too. According to the notations used in the proof of Theorem 1, for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$, $\eta_\sigma(S^{\mathscr{G}})$ is the fixed point of $\bar{c}\big(\eta_\sigma(S),\overset{(\sigma)}{\mathscr{G}}\big)$ and $\eta_\sigma(T^{\mathscr{G}})$ is the fixed point of $\bar{c}\big(\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}}\big)$, given by Lemma 1. In virtue of the second assertion of this lemma, $\eta_\sigma(S^{\mathscr{G}})+\eta_\sigma(T^{\mathscr{G}})=$ $=\eta_\sigma(S^{\mathscr{G}}+T^{\mathscr{G}})$ is the fixed point of $\bar{c}\big(\eta_\sigma(S)+\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}}\big)=\bar{c}\big(\eta_\sigma(S+T),\overset{(\sigma)}{\mathscr{G}}\big)$ for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$. This means that $S^{\mathscr{G}}+T^{\mathscr{G}}\in\mathbf{A}(S+T)=\mathscr{K}(S+T,\mathscr{G})\cap\mathbf{A}^{\mathscr{G}}$. Thus $S^{\mathscr{G}}+T^{\mathscr{G}}=(S+T)^{\mathscr{G}}$. It is evident that $T\to T^{\mathscr{G}}$ is homogenous. Now if $T\in\mathbf{A}^+$, then $T^{\mathscr{G}}\geqq 0$ as $T^{\mathscr{G}}\in\mathscr{K}(T,\mathscr{G})\subseteqq\mathbf{A}^+$. If $T\in\mathbf{A}^+$ and $T\neq O$, then $T^{\mathscr{G}}\neq O$. Indeed, if $T^{\mathscr{G}}=O$ then, in virtue of (i), we have $\sigma(T)=\sigma(T^{\mathscr{G}})=0$ for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$. Since $\mathbf{A}$ is $\mathscr{G}$-finite, from this it follows $T=O$, which completes the proof of (ii).

(iii) follows easily from the construction of the mapping $T\to T^{\mathscr{G}}$.

(iv) First we prove that the mapping $T\to T^{\mathscr{G}}$ is *normal* that is if $\{T_\iota\}_{\iota\in I}$ is an upward directed family of elements of $\mathbf{A}^+$ with $\sup\limits_{\iota\in I}T_\iota=T$, then $\sup\limits_{\iota\in I}T_\iota^{\mathscr{G}}=T^{\mathscr{G}}$ holds. In fact, since $T\to T^{\mathscr{G}}$ is positive, $\{T_\iota^{\mathscr{G}}\}$ is an upward directed family of $(\mathbf{A}^{\mathscr{G}})^+$ and $T_\iota^{\mathscr{G}}\leqq T^{\mathscr{G}}$ $(\iota\in I)$. Put $S=\sup_{\iota\in I}T_\iota^{\mathscr{G}}$. Then $S\in\mathbf{A}^{\mathscr{G}}$ ([3], App. II.), and $S\leqq T^{\mathscr{G}}$. In virtue of (i), for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ we obtain that

$$\sigma(T^{\mathscr{G}}-S)=\sigma(T^{\mathscr{G}})-\sigma(S)=\sigma(T)-\sup_{\iota\in I}\sigma(T_\iota^{\mathscr{G}})=$$
$$=\sigma(T)-\sup_{\iota\in I}\sigma(T_\iota)=\sigma(T)-\sigma(T)=0.$$

So $T^{\mathscr{G}}=S=\sup T_\iota^{\mathscr{G}}$. From this it follows that $T\to T^{\mathscr{G}}$ is ultra-weakly continuous ([3], chap. I, § 4, Th. 2). Furthermore, for every $T\in\mathbf{A}$ we obtain

$$O\leqq[(T-T^{\mathscr{G}})^*(T-T^{\mathscr{G}})]^{\mathscr{G}}=(T^*T)^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}+T^{*\mathscr{G}}T^{\mathscr{G}}=$$
$$=(T^*T)^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}$$

(cf. (ii) and (iii)). Thus $T^{*\mathscr{G}}T^{\mathscr{G}}\leqq(T^*T)^{\mathscr{G}}$, and this gives that $T\to T^{\mathscr{G}}$ is ultra-strongly continuous as well ([3]. chap. I, § 4, Th. 2).

(v) is evident.

(vi) is a consequence of the fact that $\mathscr{K}(\theta(T),\mathscr{G})=\mathscr{K}(T,\mathscr{G})$ for every $T\in\mathbf{A}$. Hence the first part of Theorem 2 is proved.

As far as the second part of Theorem 2 is concerned, we can proceed as follows. Let T_0 be an arbitrary element of $(\mathbf{A}^{\mathscr{G}})^+$ such that $T_0 \neq O$. Then there exists an element x of $\mathfrak{H}$ such that $(T_0 x|x) > 0$. For every $T \in \mathbf{A}$ put

$$\sigma(T) = (T'x|x). \tag{3. 1}$$

By our hypotheses on the mapping $T \to T'$, one can easily see that $\sigma \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with $\sigma(T_0) \neq 0$. Thus, in virtue of Definition 1, $\mathbf{A}$ is $\mathscr{G}$-finite. Furthermore, if $T \in \mathbf{A}$, then for every $S \in \mathscr{K}_0(T, \mathscr{G})$ we get that $S' = T'$ (cf. especially hypothesis b) in Theorem 2). As $T \to T'$ is supposed to be ultra-weakly continuous, the same holds for every $S \in \mathscr{K}(T, \mathscr{G})$. In particular $T' = (T^{\mathscr{G}})' = T^{\mathscr{G}}$, which completes the proof of Theorem 2.

Definition 2. If the von Neumann algebra $\mathbf{A}$ is finite with respect to a group $\mathscr{G}$ of its automorphisms, then the mapping $T \to T^{\mathscr{G}}$ given in Theorem 2 is called the *$\mathscr{G}$-canonical mapping* of $\mathbf{A}$.

§ 4

1. Let us give some direct consequences of the results of § § 2—3.

Proposition 3. *Let $\mathbf{A}$ be a von Neumann algebra, and let $\mathscr{G}$ be a group of automorphisms of $\mathbf{A}$. Suppose that $\mathbf{A}$ is $\mathscr{G}$-finite. If $\sigma_1, \sigma_2 \in \mathscr{R}(\mathbf{A}, \mathscr{G})$ are such that, for every $T \in \mathbf{A}^{\mathscr{G}}$, $\sigma_1(T) = \sigma_2(T)$ holds, then $\sigma_1 = \sigma_2$.*

Proof. If $T \in \mathbf{A}$ then

$$\sigma_1(T) = \sigma_1(T^{\mathscr{G}}) = \sigma_2(T^{\mathscr{G}}) = \sigma_2(T)$$

(cf. Theorem 2, (i)), where $T \to T^{\mathscr{G}}$ is the $\mathscr{G}$-canonical mapping of $\mathbf{A}$, and this proves Proposition 3.

In the following for a given pair $(\mathbf{A}, \mathscr{G})$, $\mathscr{R}(\mathbf{A}^{\mathscr{G}})$ will denote the set of all ultra-weakly continuous linear forms on $\mathbf{A}^{\mathscr{G}}$. Then under the same condition on $\mathbf{A}$ and $\mathscr{G}$ as in Proposition 3, we have

Corollary 1. *Every element σ_0 of $\mathscr{R}(\mathbf{A}^{\mathscr{G}})$ can be uniquely extended to an element σ of $\mathscr{R}(\mathbf{A}, \mathscr{G})$.*

Proof. For any $T \in \mathbf{A}$, put

$$\sigma(T) = \sigma_0(T^{\mathscr{G}}).$$

Then σ evidently belongs to $\mathscr{R}(\mathbf{A}, \mathscr{G})$ (cf. Theorem 2). The uniqueness of the extension follows now from Proposition 3.

Without making any restriction on $\mathbf{A}$ and $\mathscr{G}$ we can conclude from Proposition 3 also the following

Corollary 2. *If $\sigma_1, \sigma_2 \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with $\sigma_1(T) = \sigma_2(T)$ for every $T \in \mathbf{A}^{\mathscr{G}}$, then $\sigma_1 = \sigma_2$.*

Proof. Consider the projection $E = \sup(E_{\sigma_1}, E_{\sigma_2})$. It is evident that $E \in \mathbf{A}^{\mathscr{G}}$. Consider the von Neumann algebra $\mathbf{A}_E$ ([3], chap. I, § 1, no. 2). Then $\mathscr{G}$ canonically induces a group of automorphisms $\mathscr{G}_E$ of $\mathbf{A}_E$, and the restrictions σ_{1_E} and σ_{2_E} of σ_1 and σ_2 to $\mathbf{A}_E$, respectively, belong to $\mathscr{R}^+(\mathbf{A}_E, \mathscr{G}_E)$. Hence $\mathbf{A}_E$ is $\mathscr{G}_E$-finite. Furthermore, for every $T_E \in (\mathbf{A}_E)^{\mathscr{G}_E}$ we have $\sigma_{1_E}(T_E)=\sigma_{2_E}(T_E)$. So, in virtue of Proposition 3, $\sigma_{1_E}=\sigma_{2_E}$. Therefore, if $T\in\mathbf{A}$, then $\sigma_1(ETE)=\sigma_{1_E}(T_E)=\sigma_{2_E}(T_E)=\sigma_2(ETE)$. On the other hand, since $\sigma_i(T)=\sigma_i(ETE)$ $(i=1, 2)$ for every $T\in\mathbf{A}$, we can conclude that $\sigma_1=\sigma_2$, which proves Corollary 2.

Proposition 4. *Let* **A** *be a von Neumann algebra in a Hilbert space* $\mathfrak{H}$, *and let* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *be two groups of automorphisms of* **A**. *Suppose that* **A** *is* $\mathscr{G}_1$-*finite, and suppose that for every* $\theta_2\in\mathscr{G}_2$ *and* $T\in\mathbf{A}$ *we have*

$$\theta_2(T^{\mathscr{G}_1}) = (\theta_2(T))^{\mathscr{G}_1}, \tag{3.2}$$

where $T\to T^{\mathscr{G}_1}$ *is the* $\mathscr{G}_1$-*canonical mapping of* **A**.[15]) *Denote by* $\mathscr{G}_{2,1}$ *the group of automorphisms of* $\mathbf{A}^{\mathscr{G}_1}$ *defined by* $\mathscr{G}_2$ *via* (3. 2). *Now if* $A^{\mathscr{G}_1}$ *is* $\mathscr{G}_{2,1}$-*finite then* **A** *is finite with respect to the group* $\mathscr{G}=\{\mathscr{G}_1, \mathscr{G}_2\}$ *generated by* $\mathscr{G}_1$ *and* $\mathscr{G}_2$. *Hence in this case* **A** *is* $\mathscr{G}_2$-*finite, too, and we have*

$$T^{\mathscr{G}} = (T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \qquad (T\in\mathbf{A}), \tag{3.3}$$

where $T\to T^{\mathscr{G}}$ *and* $T\to T^{\mathscr{G}_2}$ *are the corresponding* $\mathscr{G}$- *and* $\mathscr{G}_2$-*canonical mappings of* **A**, *respectively.*

Proof. It is not hard to prove that $\mathbf{A}^{\mathscr{G}}=(\mathbf{A}^{\mathscr{G}_1})^{\mathscr{G}_{2,1}}$. Let now $\sigma\in\mathscr{R}^+(\mathbf{A}^{\mathscr{G}})$ be arbitrary. Since $\mathbf{A}^{\mathscr{G}_1}$ is $\mathscr{G}_{2,1}$-finite, in virtue of Corollary 1 of Proposition 3, σ can be extended to an element σ' of $\mathscr{R}^+(\mathbf{A}^{\mathscr{G}_1}, \mathscr{G}_{2,1})$. Since **A** is $\mathscr{G}_1$-finite, in virtue of the same corollary, σ' can be extended to an element σ'' of $\mathscr{R}^+(\mathbf{A}, \mathscr{G}_1)$. Now if $T\in\mathbf{A}$ and $\theta_2\in\mathscr{G}_2$, then we have

$$\sigma''(\theta_2(T)) = \sigma''((\theta_2(T))^{\mathscr{G}_1}) = \sigma''(\theta_2(T^{\mathscr{G}_1})) = \sigma'(\theta_2(T^{\mathscr{G}_1})) =$$
$$= \sigma'(T^{\mathscr{G}_1}) = \sigma''(T^{\mathscr{G}_1}) = \sigma''(T),$$

that is $\sigma''\in\mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Hence, for every $T\in(\mathbf{A}^{\mathscr{G}})^+$, $T\neq O$ there exists an element σ of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$ such that $\sigma(T)\neq 0$, and this means that **A** is $\mathscr{G}$-finite. In particular, **A** is $\mathscr{G}_2$-finite, too. Now we are going to show that for every $T\in\mathbf{A}$

$$(T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \tag{3.4}$$

holds. Now let $T\in\mathbf{A}$ be arbitrary but fixed, and let $\{K_\iota(T)\}_{\iota\in I}$ be a net of elements of $\mathscr{K}_0(T, \mathscr{G}_2)$ such that

$$\lim_{\substack{\text{strong}\\ \iota\in I}} K_\iota(T) = T^{\mathscr{G}_2}. \tag{3.5}$$

Then

$$\lim_{\substack{\text{strong}\\ \iota\in I}} [K_\iota(T)]^{\mathscr{G}_1} = (T^{\mathscr{G}_2})^{\mathscr{G}_1}. \tag{3.6}$$

(cf. Theorem 2, (iv)). On the other hand, in virtue of (3. 2) we get that

$$[K_\iota(T)]^{\mathscr{G}_1} = K_\iota(T^{\mathscr{G}_1}). \tag{3.7}$$

Thus, in virtue of (3. 6) we have

$$\lim_{\substack{\text{strong}\\ \iota\in I}} K_\iota(T^{\mathscr{G}_1}) = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \tag{3.8}$$

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[15]) Condition (3. 2) is fulfilled for instance if every element of $\mathscr{G}_1$ commutes with every element of $\mathscr{G}_2$. In fact, to show this it is enough to take into account the construction of $T^{\mathscr{G}_1}$ and the continuity properties of the elements of $\mathscr{G}_2$.

This means that $(T^{\mathcal{G}_2})^{\mathcal{G}_1}$ belongs to $\mathcal{K}(T^{\mathcal{G}_1}, \mathcal{G}_2)$, and for every $\theta_2 \in \mathcal{G}_2$, we have $\theta_2((T^{\mathcal{G}_2})^{\mathcal{G}_1}) = (\theta_2(T^{\mathcal{G}_2}))^{\mathcal{G}_1} = (T^{\mathcal{G}_2})^{\mathcal{G}_1}$ (cf. (3. 2)) and this means that $(T^{\mathcal{G}_2})^{\mathcal{G}_1} \in \mathbf{A}^{\mathcal{G}_2} \cap \cap \mathcal{K}(T^{\mathcal{G}_1}, \mathcal{G}_2)$, that is

$$(T^{\mathcal{G}_2})^{\mathcal{G}_1} = (T^{\mathcal{G}_1})^{\mathcal{G}_2}.$$

Hence (3. 4) is proved. Now it is not hard to see that the mapping

$$T \to (T^{\mathcal{G}_1})^{\mathcal{G}_2} = (T^{\mathcal{G}_2})^{\mathcal{G}_1}$$

possesses all the properties of the mapping $T \to T^{\mathcal{G}}$. Thus, by the uniqueness part of Theorem 2, we get that

$$T^{\mathcal{G}} = (T^{\mathcal{G}_1})^{\mathcal{G}_2} = (T^{\mathcal{G}_2})^{\mathcal{G}_1},$$

which proves Proposition 4.

We think it is worth formulating Theorem 1 and Theorem 2 in the following well-known particular case (cf. [3], chap. III, § 4, Th. 3; § 5, Ex. 1).

Corollary to Theorems 1 and 2. *Let* $\mathbf{A}$ *be a finite von Neumann algebra, and denote by* $\mathbf{A}^{\natural}$ *its center. Then for every* $T \in \mathbf{A}$, *the set* $\mathbf{A}^{\natural} \cap \mathcal{K}(T, \mathcal{I}(\mathbf{A}))$ *consists of one element alone. Denote it by* $T^{\natural}$. *The mapping* $T \to T^{\natural}$ *has the following properties:*

(i) *for every* $T \in \mathbf{A}$ *and for every finite normal trace* ([3], chap. I, § 6, Def. 1) φ *on* $\mathbf{A}$ *we have* $\varphi(T^{\natural}) = \varphi(T)$,

(ii) $T \to T^{\natural}$ *is strictly positive and linear;*

(iii) $T \to T^{\natural}$ *is ultra-strongly and ultra-weakly continuous;*

(iv) *if* $T \in \mathbf{A}$ *and* U *is unitary in* $\mathbf{A}$ *then* $(U^*TU)^{\natural} = T^{\natural}$ *holds;*

(v) *if* $S \in \mathbf{A}^{\natural}$ *then* $S^{\natural} = S$;

(vi) *if* $S \in \mathbf{A}^{\natural}$ *and* $T \in \mathbf{A}$ *then* $(ST)^{\natural} = ST^{\natural}$.

Conversely, if there exists a positive normal linear mapping $T \to T'$ *of* $\mathbf{A}$ *onto* $\mathbf{A}^{\natural}$ *having properties analogous to* (iv) *and* (v), *then* $\mathbf{A}$ *is finite and* $T' = T^{\natural}$ *for every* $T \in \mathbf{A}$.

Proof. In Theorems 1 and 2 take $\mathcal{I}(\mathbf{A})$ for $\mathcal{G}$.

2. Let $\mathbf{A}$ be a von Neumann algebra in a Hilbert space $\mathfrak{H}$. Denote by $\mathbf{A}_U$ the group of all unitary elements of $\mathbf{A}$. Let $U \in \mathbf{A}_U$ be an arbitrary but fixed element of $\mathbf{A}_U$. For every $T \in \mathbf{L}(\mathfrak{H})$ [16]) put

$$T \to \theta_U(T) = U^*TU.$$

The set $\mathcal{G}(\mathbf{A}_U)$ of all possible θ_U is a group of automorphisms of $\mathbf{L}(\mathfrak{H})$. In the following we are going to characterize the von Neumann algebras $\mathbf{A}$ such that $\mathbf{L}(\mathfrak{H})$ is finite with respect to $\mathcal{G}(\mathbf{A}_U)$.

Proposition 5. *Let* $\mathbf{A}$ *be a von Neumann algebra in a Hilbert space* $\mathfrak{H}$. *Then* $\mathbf{L}(\mathfrak{H})$ *is* $\mathcal{G}(\mathbf{A}_U)$*-finite if and only if* $\mathbf{A}$ *is a product*[17]) *of finite discrete factors.*[18])

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[16]) $\mathbf{L}(\mathfrak{H})$ denotes the von Neumann algebra of all bounded linear operators of $\mathfrak{H}$.

[17]) Cf. [3], chap. I, § 2, no. 2.

[18]) Cf. [3], chap. I, § 8, no. 4.

Proof. Suppose that $\mathbf{A}$ is the product of the finite discrete factors $\mathbf{M}_\iota$ $(\iota \in I)$ that is

$$\mathbf{A} = \prod_{\iota \in I} \mathbf{M}_\iota .$$

It is evident that $(U_\iota)_{\iota \in I} \in \mathbf{A}_U$ if and only if $U_\iota \in (\mathbf{M}_\iota)_U$ [19]) for every $\iota \in I$. Furthermore, for every $\iota \in I$, the group $(\mathbf{M}_\iota)_U$ is compact in the weak operator topology. Thus, using the Tychonoff theorem on the topological product of compact spaces, it is not hard to see that $\mathbf{A}_U$ is compact in the weak topology. Denote by $\lambda(dU)$ the normalized Haar measure of $\mathbf{A}_U$, and let $T \in \mathbf{L}(\mathfrak{H})$ be arbitrary. If x is any element of $\mathfrak{H}$, the function

$$U \to f_{x,T}(U) = (U^*TUx|x)$$

is continuous on $\mathbf{A}_U$, since the weak and the strong topology coincide on $\mathbf{A}_U$. So

$$\int_{\mathbf{A}_U} f_{x,T}(U)\,\lambda(dU)$$

exists. Let $x \in \mathfrak{H}$ be fixed, and for every $T \in \mathbf{L}(\mathfrak{H})$ set

$$\sigma_x(T) = \int_{\mathbf{A}_U} f_{x,T}(U)\,\lambda(dU).$$

Using the unimodularity of λ and the properties of the integral, it is easy to show that $\sigma_x \in \mathscr{R}^+(\mathbf{L}(\mathfrak{H}), \mathscr{G}(\mathbf{A}_U))$. Now if $T \in \mathbf{L}^+(\mathfrak{H})$, $T \neq 0$ then there exists an element x_0 of $\mathfrak{H}$ such that $(Tx_0|x_0) > 0$. Then $\sigma_{x_0}(T) \neq O$, which proves that $\mathbf{L}(\mathfrak{H})$ is $\mathscr{G}(\mathbf{A}_U)$-finite.

Now suppose that $\mathbf{L}(\mathfrak{H})$ is $\mathscr{G}(\mathbf{A}_U)$-finite, and let $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ be the $\mathscr{G}(\mathbf{A}_U)$-canonical mapping of $\mathbf{L}(\mathfrak{H})$ onto $\mathbf{L}(\mathfrak{H})^{\mathscr{G}(\mathbf{A}_U)}$ (cf. Theorem 2) which is equal to the commutant $\mathbf{A}'$ of $\mathbf{A}$. Let $\mathrm{Tr}(\cdot)$ be the canonical trace of $\mathbf{L}(\mathfrak{H})$ ([3], chap. I, § 6, no. 6), and let $S \in (\mathbf{A}')^+$, $S \neq O$ be arbitrary. Then there exists an element S_1 of $\mathbf{L}(\mathfrak{H})$ such that $O \leqq S_1 \leqq S$, $S_1 \neq O$, and $\mathrm{Tr}\,(S_1) < +\infty$. By the properties of the mapping $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ we obtain that $O \leqq S_1^{\mathscr{G}(\mathbf{A}_U)} \leqq S^{\mathscr{G}(\mathbf{A}_U)} = S$. Furthermore, as $\mathrm{Tr}\,(\cdot)$ is lower semicontinuous in the weak topology ([3], chap. I, § 6, Prop. 2, Cor.) and $S_1^{\mathscr{G}(\mathbf{A}_U)} \in \mathscr{K}(S_1, \mathscr{G}(\mathbf{A}_U))$, we get that $\mathrm{Tr}\,(S_1^{\mathscr{G}(\mathbf{A}_U)}) \leqq \mathrm{Tr}\,(S_1)$. On the other hand, $S_1^{\mathscr{G}(\mathbf{A}_U)} \neq O$ since the mapping $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ is strictly positive. So we have proved that for every $S \in (\mathbf{A}')^+$, $S \neq O$ there exists an element $S' \in (\mathbf{A}')^+$, $S' \neq O$, $S' \leqq S$ such that $\mathrm{Tr}\,(S') < +\infty$. Now let $E \neq O$ be a projection in $\mathbf{A}'$. Then there exists a non-zero element R of $(\mathbf{A}')^+$ with $R \leqq E$ and $\mathrm{Tr}\,(R) < +\infty$. Let $R = \int \lambda\, dF_\lambda$ be the spectral representation of R and set $F = I - F_{\frac{\|R\|}{2}+0}$. Then it is evident that $F \in \mathbf{A}'$, $F \neq O$ and $\frac{\|R\|}{2} F \leqq R$. Therefore, $\mathrm{Tr}(F) < +\infty$. Furthermore, as F is a projection, we obtain that $F \leqq E$. Let now F_0 be any of the projections of $\mathbf{A}'$ such that $F_0 \neq O$, $F_0 \leqq E$ and $\mathrm{Tr}\,(F_0)$ is minimal. Then F_0 is minimal in $\mathbf{A}'$. Indeed, $F_0' \in \mathbf{A}'$, $F_0' \neq O$, $F_0' \neq F_0$, $F_0' \leqq F_0$ would imply $F_0' \leqq E$, $\mathrm{Tr}\,(F_0') < +\infty$ and $\mathrm{Tr}(F_0') < \mathrm{Tr}\,(F_0)$ which contradicts the minimality of $\mathrm{Tr}\,(F_0)$. Thus, every non-zero projection of $\mathbf{A}'$ majorizes a non-zero minimal projection of $\mathbf{A}'$. Hence, in virtue

[19]) $(\mathbf{M}_\iota)_U$ denotes the group of the unitary elements of $\mathbf{M}_\iota$.

of Ex. 4, p. 126 of [3], **A**′ and so **A** is a product of discrete factors. Since **A** is finite, each factor occuring in the decomposition of **A** is finite ([3], chap. I, § 8, no. 2). Thus the proof of Proposition 5 is comlete.

Corollary. *In order that the group* $\mathbf{A}_U$ *of the unitary elements of a von Neumann algebra* **A** *be compact in the weak topology, it is necessary and sufficient that* **A** *be the product of finite discrete factors.*

Proof. The sufficiency of our condition is evident by the Tychonoff theorem (cf. the first step of the proof of Proposition 5). Now, if $\mathbf{A}_U$ is weakly compact, then arguing in the same way as in the proof of Proposition 5, we obtain that $\mathbf{L}(\mathfrak{H})$ is $\mathcal{G}(\mathbf{A}_U)$-finite which means, by Proposition 5, that **A** is a product of finite discrete factors. Hence the proof of Corollary is complete.

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