# A convergence theorem of orthogonal series

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## Introduction

Let $\varphi_1(x), \varphi_2(x), \ldots$ be an orthonormal sequence defined on a measure space $\{X, S, \mu\}$. For the sake of simplicity we assume that $\mu(X)=1$. Further let $c_1, c_2, \ldots$ be a sequence of real numbers with

(1) $$\sum_{i=1}^{\infty} c_i^2 < \infty.$$

A fundamental problem of the theory of orthogonal series is to find conditions implying the almost everywhere convergence of the series

(2) $$\sum_{i=1}^{\infty} c_i \varphi_i(x).$$

In general the condition (1) does not imply the almost everywhere convergence of the series (2). However, the classical Menšov—Rademacher theorem states:

Theorem A. *If*

(3) $$\sum_{i=1}^{\infty} c_i^2 \log^2 i < \infty$$

*then the series* (2) *is convergent almost everywhere.*

Under certain special restrictions on the sequence $\{\varphi_k(x)\}$ the condition (3) can be replaced by weaker ones. For example the classical Kolmogorov theorem states that *if the functions* $\varphi_1(x), \varphi_2(x), \ldots$ *are independent in the sense of probability theory, with expectation* 0 *and variance* 1, *then condition* (1) *implies the almost everywhere convergence of* (2). A similar result is due to G. ALEXITS [1]. He introduced the following definitions:

Definition 1. The sequence $\varphi_1(x), \varphi_2(x), \ldots$ of measurable functions is called a multiplicative system if

$$\int_X \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_k} \, d\mu = 0 \qquad (i_1 < i_2 < \ldots < i_k; \quad k = 1, 2, \ldots).$$

Definition 2. The sequence $\varphi_1(x), \varphi_2(x), \dots$ of measurable functions is called a strongly multiplicative system if the system $\{\varphi_{i_1}\varphi_{i_2} \dots \varphi_{i_k}\}$ is an orthogonal system, i.e. if

$$\int_X \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \dots \varphi_{i_k}^{\alpha_k} \, d\mu = 0 \qquad (i_1 < i_2 < \dots < i_k; \quad k = 1, 2, \dots)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2 but at least one element of the sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ is equal to 1.

Definition 3. The sequence $\varphi_1(x), \varphi_2(x), \dots$ of measurable functions is called an equinormed strongly multiplicative system (ESMS) if

$$\int_X \varphi_i \, d\mu = 0, \quad \int_X \varphi_i^2 \, d\mu = 1 \qquad (i = 1, 2, \dots),$$

(4)

$$\int_X \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \dots \varphi_{i_k}^{\alpha_k} \, d\mu = \int_X \varphi_{i_1}^{\alpha_1} \, d\mu \int_X \varphi_{i_2}^{\alpha_2} \, d\mu \dots \int_X \varphi_{i_k}^{\alpha_k} \, d\mu \qquad (i_1 < \dots < i_k; \quad k = 1, 2, \dots)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2.

Making use of these definitions, ALEXITS and TANDORI ([2]) proved the following

Theorem B. *If $\varphi_1(x), \varphi_2(x), \dots$ is a uniformly bounded ESMS, then condition* (1) *implies the almost everywhere convergence of the series* (2).

Obviously any independent system with $\int_X \varphi_i d\mu = 0$, $\int_X \varphi_i^2 d\mu = 1$ is an ESMS, therefore ALEXITS's theorem would be much stronger the KOLMOGOROV's if the condition of boundedness could be dropped. A previous paper ([3]) of the author shows that there are some further theorems (the central limit theorem and the law of the iterated logarithm) known to hold for independent random variables which remain valid for ESMS.

## § 1. The Theorem

The aim of this paper is to prove the following

Theorem 1. *Let $\varphi_1(x), \varphi_2(x), \dots$ be a sequence of measurable functions defined on a measure space $\{X, S, \mu\}$ with $\mu(X) = 1$. Suppose that*

(5)
$$\int_X \varphi_i^4 \, d\mu \leqq K \qquad (i = 1, 2, \dots)$$

*and*

(6)
$$\int_X \varphi_i^2 \varphi_j \varphi_k \, d\mu = \int_X \varphi_i^2 \varphi_j \, d\mu = \int_X \varphi_i \varphi_j \varphi_k \varphi_l \, d\mu = \\ = \int_X \varphi_i \varphi_j \varphi_k \, d\mu = \int_X \varphi_i \varphi_j \, d\mu = \int_X \varphi_i \, d\mu = 0,$$

*where the indices i, j, k, l are different, and K is a positive constant. Further let $c_1, c_2, \ldots$ be a sequence of real numbers and suppose that there exists an integer r (depending on $\{c_k\}$) such that*

(7)
$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

*where* [1])

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if} \quad x \geqq 2 \\ 1 & \text{if} \quad 0 < x < 2 \end{cases}$$

*and $l_r(x)$ is the r-th iterate of $l(x)$ i.e. $l_r(x) = l(l_{r-1}(x))$. Then the series*

$$\sum_{k=1}^{\infty} c_k \varphi_k(x)$$

*is convergent almost everywhere.*

Remark 1. If $\{\varphi_k\}$ is a sequence of fourwise independent random variables with expectation 0 and variance 1 and with uniformly bounded fourth moments then (5) and (6) hold.

Remark 2. Condition (7) is not very far from condition (1). This facts suggests the conjecture that (7) can be replaced by (1).

The proof of this theorem is based on three lemmas.

Lemma 1. *If $\varphi_1, \varphi_2, \ldots$ is a sequence of measurable functions for which* (5) *and* (6) *hold, then*

(8)
$$\int_X \max_{1 \leqq k \leqq n} |c_1 \varphi_1 + c_2 \varphi_2 + \ldots + c_k \varphi_k|^4 \, d\mu \leqq 8Kl^4(n) \left( \sum_{j=1}^{n} c_j^2 \right)^2$$

*where $c_1, c_2, \ldots, c_n$ is an arbitrary sequence of real numbers.*

Remark 3. This lemma is not the best possible. In [3] it was proved that in the case $c_1 = c_2 = \ldots = c_n = 1$, $l^4(n)$ can be replaced by $O(1)l^3(n)$. The same method can be applied in this more general case to obtain a stronger inequality. Unfortunately using such a stronger inequality instead of (8) we cannot obtain a stronger result than Theorem 1, therefore we do not intend to attain the best possible inequality.

Lemma 2. *If $c_1, c_2, \ldots$ is a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

*then there exists a sequence $n_1, n_2, \ldots$ of integers for which*

(9)
$$\sum_{k=1}^{\infty} \left( \sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right) l_{r-1}^2(k) < \infty,$$

(10)
$$\sum_{k=1}^{\infty} \left( \sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right)^2 l^4(n_{k+1} - n_k) < \infty.$$

[1]) $\log x$ means the logarithm with the base 2.

Lemma 3. *If $\{\varphi_k\}$ is a sequence of measurable functions for which* (5) *and* (6) *hold and $m_1 < m_2 < \dots$ is a sequence of integers then for the sequence*

(11)
$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \displaystyle\sum_{j=m_k+1}^{m_{k+1}} c_j \varphi_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

*where*
$$\alpha_k = \left[ \sum_{j=m_k+1}^{m_{k+1}} c_j^2 \right]^{1/2},$$

*we have*
$$\int_X \psi_k^4 \, d\mu \leqq 4K$$

*and* (6).

## § 2. The proof

Proof of Lemma 1. First of all we assume that $n = 2^\nu$ $(\nu = 1, 2, \dots)$ and introduce the following notations

$$\sigma_j = c_1\varphi_1 + c_2\varphi_2 + \dots + c_j\varphi_j \qquad (j = 1, 2, \dots),$$
$$\psi_{\alpha\beta} = c_{\alpha+1}\varphi_{\alpha+1} + c_{\alpha+2}\varphi_{\alpha+2} + \dots + c_\beta\varphi_\beta$$

where $\alpha = \mu 2^k$; $\beta = (\mu+1)2^k$; $\mu = 0, 1, 2, \dots, 2^{\nu-k} - 1$; $k = 0, 1, 2, \dots, \nu - 1$. Consider the function $\sigma_j$ as the sum of some $\psi_{\alpha\beta}$. Let us put

$$\sigma_j = \sum_i \psi_{\alpha_i \beta_i}$$

where $\beta_1 - \alpha_1 > \beta_2 - \alpha_2 > \dots$. Clearly the number of the members of the sum $\sum_i \psi_{\alpha_i \beta_i}$ is less than $\nu$. Therefore by the Schwarz inequality we have

$$\sigma_j^4 = \left( \sum_i \psi_{\alpha_i \beta_i} \right)^4 \leqq \nu^2 \left( \sum_i \psi_{\alpha_i \beta_i}^2 \right)^2 \leqq \nu^3 \sum_i \psi_{\alpha_i \beta_i}^4$$

which implies

(12)
$$\int_X \max_{1 \leqq j \leqq 2^\nu} \sigma_j^4 \, d\mu \leqq \nu^3 \sum_{\alpha, \beta} \int_X \psi_{\alpha\beta}^4 \, d\mu$$

where $\alpha$ and $\beta$ run over all their possible values.

We obtain an estimation of the right hand side of (12) as follows

(13)
$$\int_X \psi_{\alpha\beta}^4 \, d\mu = \sum_{j=\alpha+1}^{\beta} c_j^4 \int_X \varphi_j^4 \, d\mu + 6 \sum_{\alpha < i < j \leqq \beta} c_i^2 c_j^2 \int_X \varphi_i^2 \varphi_j^2 \, d\mu +$$
$$+ 4 \sum_{\substack{\alpha < i, j \leqq \beta \\ i \neq j}} c_i^3 c_j \int_X \varphi_i^3 \varphi_j \, d\mu \leqq K \left\{ \sum_{j=\alpha+1}^{\beta} c_j^4 + 6 \sum_{\alpha < i < j \leqq \beta} c_i^2 c_j^2 + 4 \sum_{\substack{\alpha < i, j \leqq \beta \\ i \neq j}} |c_i^3 c_j| \right\}.$$

Summing the right hand side of (13) for each $\alpha, \beta$ we obtain any member of it at most $\nu$ times, so we have

$$\sum_{\alpha,\beta} \int_X \psi_{\alpha\beta}^4 \, d\mu \leqq \nu K \left\{ \sum_{j=1}^{2^\nu} c_j^4 + 6 \sum_{1 \leqq i < j \leqq 2^\nu} c_i^2 c_j^2 + 4 \sum_{\substack{1 \leqq i, j \leqq 2^\nu \\ i \neq j}} |c_i^3 c_j| \right\}.$$

It is easy to see that

$$\sum_{\substack{1 \leqq i, j \leqq 2^\nu \\ i \neq j}} |c_i^3 c_j| \leqq \frac{1}{2} \sum_{j=1}^{2^\nu} c_j^4 + \sum_{1 \leqq i < j \leqq 2^\nu} c_i^2 c_j^2 .$$

Hence we have

$$\sum_{\alpha,\beta} \int_X \psi_{\alpha\beta}^4 \, d\mu \leqq 4\nu K \left( \sum_{j=1}^{2^\nu} c_j^2 \right)^2.$$

Thus in the case $n = 2^\nu$ we obtained

$$\int_X \sup_{1 \leqq k \leqq n} |c_1 \varphi_1 + c_2 \varphi_2 + \ldots + c_k \varphi_k|^4 \, d\mu \leqq 4K l^4(n) \left( \sum_{j=1}^{n} c_j^2 \right)^2.$$

Our inequality in the case $2^\nu \leqq n < 2^{\nu+1}$ follows immediately from this fact, setting

$$c_{n+1} = c_{n+2} = \ldots = c_{2^{\nu+1}} = 0$$

and using the inequality

$$2 \log^4 n \geqq (\log 2n)^4 \geqq (\nu + 1)^4$$

if $n$ is large enough.

Proof of Lemma 2. Set

$$A = \sum_{k=1}^{\infty} c_k^2 \, l_r^2(k)$$

then

$$A \geqq \sum_{k=n}^{\infty} c_k^2 \, l_r^2(k) \geqq l_r^2(n) \sum_{k=n}^{\infty} c_k^2$$

and

$$\sum_{k=n}^{\infty} c_k^2 \leqq \frac{A}{l_r^2(n)} .$$

Therefore we have

$$\sum_{k=2^\nu+1}^{2^{\nu+1}} c_k^2 \leqq \frac{A}{l_{r-1}^2(\nu)} . \tag{14}$$

Now we can find between $2^\nu + 1$ and $2^{\nu+1}$ a sequence of integers

$$2^\nu + 1 = \tau_0^{(\nu)} \leqq \tau_1^{(\nu)} \leqq \ldots \leqq \tau_{s_\nu - 1}^{(\nu)} \leqq \tau_{s_\nu}^{(\nu)} = 2^{\nu+1}$$

as follows: Let $\tau_2^{(\nu)}$ be the smallest integer for which

$$\sum_{j=2^\nu+1}^{\tau_2^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 \, l_{r-1}^2(\nu)}$$

and let $\tau_1^{(\nu)} = \tau_2^{(\nu)} - 1$. Similarly let $\tau_4^{(\nu)}$ be the smallest integer for which

$$\sum_{j=\tau_2^{(\nu)}+1}^{\tau_4^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

and let $\tau_3^{(\nu)} = \tau_4^{(\nu)} - 1$. In general, if $\tau_{2l}^{(\nu)}$ is defined, we define $\tau_{2(l+1)}^{(\nu)}$ as the smallest integer for which

$$\sum_{j=\tau_{2l}^{(\nu)}+1}^{\tau_{2(l+1)}^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

and let $\tau_{2l+1}^{(\nu)} = \tau_{2(l+1)}^{(\nu)} - 1$. Now let

$$2^\nu + 1 = t_0^{(\nu)} < t_1^{(\nu)} < \ldots < t_{p_\nu}^{(\nu)} = 2^{\nu+1}$$

be the different elements of the sequence $\tau_0^{(\nu)}, \tau_1^{(\nu)}, \ldots, \tau_{s_\nu}^{(\nu)}$. Clearly

$$p_\nu \leqq 2\nu^6.$$

Define now the sequence $\{n_k\}$ as the union of the sequences $t_0^{(\nu)}, t_1^{(\nu)}, \ldots, t_{p_\nu}^{(\nu)}$ i.e. the sequence $n_1, n_2, \ldots$ is the same as the sequence

$$t_0^{(1)}, t_1^{(1)}, t_0^{(2)}, t_1^{(2)}, t_2^{(2)}, t_0^{(3)}, t_1^{(3)}, \ldots, t_{p_3}^{(3)}, t_0^{(4)}, t_1^{(4)}, \ldots, t_{p_4}^{(4)}, t_0^{(5)}, t_1^{(5)}, \ldots, t_{p_5}^{(5)}, \ldots .$$

Clearly if $n_k \in (2^\nu, 2^{\nu+1}]$ then $k \leqq 2\sum_{j=1}^{\nu} j^6 \leqq 2(\nu+1)^7$. We prove that (9) and (10) hold for this sequence $\{n_k\}$. We have

$$A = \sum_{j=1}^{\infty} c_j^2 l_r^2(j) = \sum_{\nu=1}^{\infty} \sum_{j=2^\nu+1}^{2^{\nu+1}} c_j^2 l_r^2(j) \geqq \frac{1}{2} \sum_{\nu=1}^{\infty} \sum_{j=2^\nu+1}^{2^{\nu+1}} c_j^2 l_{r-1}^2(\nu+1) =$$

$$= \frac{1}{2} \sum_{\nu=1}^{\infty} \sum_{n_k \in (2^\nu, 2^{\nu+1}]} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2(\nu+1) \geqq$$

$$\geqq \frac{1}{4} \sum_{\nu=1}^{\infty} \sum_{n_k \in (2^\nu, 2^{\nu+1}]} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2(2(\nu+1)^7) \geqq \frac{1}{4} \sum_{k=1}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2(k)$$

that proves (9).

If $n_k \in (2^\nu, 2^{\nu+1}]$ then by the definition of $\{n_k\}$ we have

$$n_{k+1} - n_k \leqq 2^{\nu+1}$$

and either

$$\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \leqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

or

$$n_{k+1} - n_k = 1;$$

this gives (10).

Proof of Lemma 3 is so simple that we can omit it.

Proof of Theorem 1. First of all we prove that the series (2) is convergent almost everywhere if

$$\sum_{k=1}^{\infty} c_k^2 \, l_2^2(k) < \infty. \tag{15}$$

Let $\{n_k\}$ be a sequence of integers for which

$$\sum_{k=1}^{\infty} \left( \sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right) l^2(k) < \infty \tag{16}$$

and (10) holds. Set

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \displaystyle\sum_{j=n_k+1}^{n_{k+1}} c_j \varphi_j & \text{if } \alpha_k > 0 \\ 0 & \text{if } \alpha_k = 0 \end{cases}$$

where $\alpha_k = \left[ \sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right]^{1/2}$ and put $\sigma_N = \sum_{j=1}^{N} c_j \varphi_j$. Clearly we have

$$\sigma_{n_K} = \sum_{k=1}^{K-1} \alpha_k \psi_k. \tag{17}$$

By Lemma 3, Theorem A and (16), the sequence $\{\sigma_{n_K}\}$ is convergent almost everywhere. By Lemma 1 and (10)

$$\sum_{k=1}^{\infty} \int_X \max_{n_k < j \leq n_{k+1}} \left( \sum_{l=n_k+1}^{j} c_l \varphi_l \right)^4 d\mu < \infty. \tag{18}$$

Hence by the Beppo Levi theorem we have

$$\sum_{k=1}^{\infty} \max_{n_k < j \leq n_{k+1}} \left( \sum_{l=n_k+1}^{j} c_l \varphi_l \right)^4 < \infty,$$

hence

$$\max_{n_k < j \leq n_{k+1}} \left| \sum_{l=n_k+1}^{j} c_l \varphi_l \right| \to 0$$

almost everywhere. This fact and the almost everywhere convergence of the sequence (17) prove our theorem in the case when (15) holds.

Now Theorem 1 can be proved by induction. Suppose that, for every sequence $\{a_k\}$ and for every system $\{\chi_k\}$ having the properties (5) and (6) we have already proved that the condition

$$\sum_{k=1}^{\infty} a_k^2 \, l_{r-1}^2(k) < \infty \tag{19}$$

implies the almost everywhere convergence of the series

$$\sum_{k=1}^{\infty} a_k \chi_k(x).$$

Let $\{c_k\}$ be a sequence of real numbers for which (7) holds. Now we can construct a sequence $\{n_k\}$ for which (9) and (10) hold. Then we can obtain by the same way that $\sigma_{n_K}$ (defined by (17)) is convergent almost everywhere, if we replace the reference to Theorem A by a reference to the condition (19) of our induction. (18) follows from (10).

## References

[1] G. ALEXITS, *Convergence problems of orthogonal series* (Budapest, 1961).
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