On interpolation of L_p spaces with weight functions

By JAAK PEETRE in Lund (Sweden)

0. Introduction

According to a classical theorem of BERNSTEIN (1914) a sufficient condition for a function f on the real line $(-\infty, \infty)$ to be given by an absolutely convergent Fourier integral,

$$f(x) = \frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\,d\xi, \quad \hat{f}\in L_1,$$

is that $f\in L_2$ and satisfies a Lipschitz condition of exponent $>\frac{1}{2}$ in L_2, i. e.

$$\|f(x+t)-f(x)\|_{L_2} = O(t^{\frac{1}{2}+\varepsilon}) \text{ for some } \varepsilon>0 \text{ and as } t\to 0.$$

A more precise condition in this sense reads:

$$\int_0^\infty \|f(x+t)-f(x)\|_{L_2}\frac{dt}{t^{3/2}}<\infty. \tag{0. 1}$$

Now the requirement $f\in L_2$ is inessential. (Usually this theorem is given for Fourier series but the change to integrals is immediate; see [9], vol. 1, pp. 240—241; see also [8].) More recently BEURLING (see [2]) has shown that condition (0. 1) on f is equivalent to the following one on $\hat{f}$:

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\omega(|\xi|)\,d\xi<\infty \quad \text{for some } \textit{non-decreasing } \omega \text{ with } \int_0^\infty \frac{d\xi}{\omega(|\xi|)}<\infty. \tag{0. 2}$$

This is of importance in some questions of Spectral Synthesis (see [1], [2]).

The purpose of this note is to show how this as well as some other results of [2] can be interpreted from the point of view of the theory of interpolation spaces. The plan is as follows. We first (Section 1) briefly summarize some general notions on interpolation spaces $(A_0, A_1)_{\theta,q}$ which will be needed in what follows. Then (Section 2) we specialize to the case of spaces $L_p(w)$, L_p space with weight function w. In particular we characterize $(L_p(w_0), L_p(w_1))_{\theta,q}$ when $q=1$ or $q=\infty$. This should be contrasted to known results when $q=p$ (see [7]). The applications to Fourier integrals finally are given Section 3.

1. General notions on interpolation spaces

Let A_0 and A_1 be two Banach spaces both continuously imbedded in the same topological vector space $\mathfrak{A}$.

If $0<t<\infty$, $f\in A_0+A_1$, set

$$K(t,f)=K(t,f;A_0,A_1)=\inf_{f=f_0+f_1}(\|f_0\|_{A_0}+t\|f_1\|_{A_1})$$

and, if $0<t<\infty$, $f\in A_0\cap A_1$,

$$J(t,f)=J(t,f;A_0,A_1)\cdot\max\left(\|f\|_{A_0},t\|f\|_{A_1}\right).$$

There are basically two ways of obtaining interpolation spaces:

1° We impose a "growth condition" on $K(t,f)$ of the form

$$\Phi[K(t,f)]<\infty.$$

where Φ is a suitable functional.

2° Upon representing f in the form

$$f=\int_0^\infty f(t)\frac{dt}{t}\quad\text{(non unique!)}$$

we impose a "growth condition" on $J(t,f(t))$ of the form

$$\Phi\left[J(t,f(t))\right]<\infty.$$

The most important special case is when

$$\Phi[\varphi]=\Phi_{\theta,q}[\varphi]=\left[\int_0^\infty (t^{-\theta}\varphi(t))^q\frac{dt}{t}\right]^{1/q},\ 0<\theta<1,\quad 1\leqq q\leqq\infty.$$

In this case the two constructions 1° and 2° lead to the same spaces (up to an equivalence of norm) which we shall denote by $(A_0,A_1)_{\theta,q}$.

For more details about these spaces, see e. g. [6], see also [3], [4], [5].

2. The case of L_p spaces with weight functions

Let Z be a locally compact space provided with a positive measure μ. Let w be a positive μ-measurable function (weight function). We denote by $L_p(w)$, $1<p<\infty$, the space of μ-measurable function f such that $|wf|^p$ is μ-integrable and endow it with the norm

$$\|f\|_{L_p(w)}=\|wf\|_{L_p}=\left(\int_Z|wf|^p\,d\mu\right)^{1/p}.$$

Let now w_0 and w_1 be any two fixed such functions and take

$$A_0 = L_p(w_0), \quad A_1 = L_p(w_1), \qquad 1<p<\infty.$$

Thus in what follows

$$K(t,f)=K(t,f;\, L_p(w_0),\, L_p(w_1)), \quad J(t,f)=J(t,f;\, L_p(w_0),\, L_p(w_1)).$$

By [7] we have

(2. 1) $$K(t,f)\sim \|f\|_{L_p(\min(w_0,\,tw_1))},$$

(2. 2) $$J(t,f)\sim \|f\|_{L_p(\max(w_0,\,tw_1))}.$$

Put further

$$K_1(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(1)}\begin{cases} = w_0 & (w_0<tw_1),\\ =0 & (w_0\geqq tw_1),\end{cases}$$

$$K_2(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(2)}\begin{cases} = 0 & (w_0<tw_1),\\ =tw_1 & (w_0\geqq tw_1),\end{cases}$$

$$K_3(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(3)}\begin{cases} \sim w_0\sim tw_1 & (w_0<tw_1<2w_0).\\ =0 & (\text{elsewhere}).\end{cases}$$

Then holds

Theorem 2. 1. *The following inequalities are valid:*

(2. 3) $$C^{-1}K_i(t,f)\leqq K(t,f)\leqq C\left\{\int_0^\infty\left[\varphi_i\left(\frac{1}{\sigma}\right)K_i(t\sigma,f)\right]^p\frac{d\sigma}{\sigma}\right\}^{1/p} \qquad (i=1,2,3)$$

where

$$\varphi_1(\sigma)\begin{cases} =\sigma & (\sigma<1),\\ =0 & (\sigma\geqq 1),\end{cases} \quad \varphi_2(\sigma)\begin{cases} =0 & (\sigma<1),\\ =1 & (\sigma\geqq 1),\end{cases}$$

$$\varphi_3(\sigma) = \min(1,\sigma)\begin{cases} =\sigma & (\sigma<1),\\ =1 & (\sigma\geqq 1).\end{cases}$$

It follows that

$$f\in(L^p(w_0),\, L^p(w_1))_{\theta,q} \qquad (q\geqq p)$$

if and only if

$$\left[\int_0^\infty [t^{-\theta}K_i(t,f]^q\frac{dt}{t}\right]^{\frac{1}{q}}<\infty \qquad (i=1,2,3).$$

Proof. For simplicity we consider the case $i=1$ only; the other two cases can be treated in a similar fashion. The inequality to the left in (2. 3)

$$K_1(t,f)\leqq CK(t,f)$$

is trivial, by (2. 1), since clearly $w^{(1)} \leqq \min(w_0, tw_1)$. Remains the inequality to the right. But we get, again by (2. 1),

$$(K(t,f))^p \leqq C^p \int_Z |\min(w_0, tw_1)f|^p \, d\mu \leqq$$

$$\leqq C^p \Big[\int_{w_0 < 2tw_1} |w_0 f|^p \, d\mu + 2^{-p} \int_{2tw_1 \leqq w_0 < 2^2 tw_1} |w_0 f|^p \, d\mu +$$

$$+ 2^{-2p} \int_{2^2 tw_1 \leqq w_0 < 2^3 tw_1} |w_0 f|^p \, d\mu + \dots \Big] \leqq$$

$$\leqq C^p [[K_1(2t,f)]^p + 2^{-p}[K_1(2^2 t,f)]^p + 2^{-2p}[K_1(2^3 t,f)]^p + \dots] \leqq$$

$$\leqq C^p \int_1^\infty \left[\frac{1}{\sigma} K_1(t\sigma, f)\right]^p \frac{d\sigma}{\sigma}$$

and this inequality too follows.

If $f \in (L_p(w_0), L_p(w_1))_{\theta,q}$ the trivial half of (2. 3) shows that

$$\left[\int_0^\infty [t^{-\theta} K_1(t,f)]^q \frac{dt}{t}\right]^{1/p} < \infty.$$

Conversely, if this condition holds and moreover $q \geqq p$, upon writing the other half of (2. 3) as

$$[t^{-\theta} K(t,f)]^p \leqq C^p \int_1^\infty \sigma^{(\theta-1)p} [(\sigma t)^{-\theta} K_1(\sigma t, f)]^p \frac{d\sigma}{\sigma}$$

we get by MINKOWSKI's inequality

$$\left[\int_0^\infty [t^{-\theta} K(t,f)]^q \frac{dt}{t}\right]^{p/q} \leqq C^p \int_1^\infty \sigma^{(\theta-1)p} \frac{d\sigma}{\sigma} \left[\int_0^\infty [t^{-\theta} K_1(t,f)]^q \frac{dt}{t}\right]^{1/q} < \infty$$

which shows $f \in (L_p(w_0), L_p(w_1))_{\theta,q}$.

Next we prove — this is our main new contribution —

Theorem 2. 2 *We have*

(2. 4) $$f \in (L_p(w_0), L_p(w_1))_{\theta,1}$$

if and only if

(2. 5) $$f \in L_p\left(w_0 \varphi\left(\frac{w_1}{w_0}\right)\right)$$

for some non-decreasing φ *such that*

$$\int_0^\infty \left(\frac{t^\theta}{\varphi(t)}\right)^{p'} \frac{dt}{t} < \infty \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

Proof i) Assume that (2. 5) holds true. Put

$$f_k = \begin{cases} f & (w_0 < 2^k w_1 \leqq 2w_0), \\ 0 & (\text{elsewhere}). \end{cases}$$

Then, by (2. 2),

$$J(2^k, f_k) \leqq C\Big[\int\limits_{w_0 < 2^k w_1 \leqq 2w_0} |w_0 f|^p \, d\mu\Big]^{1/p} \leqq$$

$$\leqq C\frac{1}{\varphi(2^{-k})}\Big[\int\limits_{w_0 < 2^k w_1 \leqq 2w_0} \Big|w_0 \varphi\Big(\frac{w_1}{w_0}\Big) f\Big|^p d\mu\Big]^{1/p},$$

the last estimate because φ is non-decreasing. It follows by HÖLDER's inequality that

$$\sum_{-\infty}^{\infty} 2^{-k\theta} J(2^k, f_k) \leqq C\|f\|_{L_p\left(w_0\varphi\left(\frac{w_1}{w_0}\right)\right)} \Big[\sum_{-\infty}^{\infty}\Big(\frac{2^{-k\theta}}{\varphi(2^{-k})}\Big)^{p'}\Big]^{1/p'} \leqq$$

$$\leqq C\|f\|_{L_p\varphi\left(w_0\varphi\left(\frac{w_1}{w_0}\right)\right)} \Big[\int\limits_0^{\infty}\Big(\frac{t^\theta}{\varphi(t)}\Big)^{p'}\frac{dt}{t}\Big]^{1/p'} < \infty.$$

If we put

$$f(t) = (\log 2)^{-1} f_k \qquad (2^k \leqq t < 2^{k+1})$$

this gives

$$\int\limits_0^{\infty} \frac{J(t, f(t))}{t^\theta}\,\frac{dt}{t} < \infty$$

and, since also $f = \int\limits_0^{\infty} f(t)\,\frac{dt}{t}$, we have established (2. 4).

ii) Conversely assume (2. 4) holds true. Then

$$\int\limits_0^{\infty} \frac{K(t, f)}{t^\theta}\,\frac{dt}{t} < \infty$$

or, in view of (2. 1),

$$\int\limits_Z \Big|w_0 \varphi\Big(\frac{w_1}{w_0}\Big) f\Big|^p d\mu < \infty$$

with

$$\varphi(\lambda) = \Big[\int\limits_0^{\infty} \frac{[\min(1, t\lambda)]^p}{t^\theta [K(t, f)]^{p-1}}\,\frac{dt}{t}\Big]^{1/p}$$

which is obviously non-decreasing. But

$$K(t, f) \leqq \max(1, t\lambda) K\Big(\frac{1}{\lambda}, f\Big).$$

Hence

$$(\varphi(\lambda))^p \geqq \frac{\lambda^\theta}{\left[K\left(\frac{1}{\lambda}, f\right)\right]^{p-1}} \int_0^\infty \frac{[\min(1, t\lambda)]^p}{(\lambda t)^\theta [\max(1, t\lambda)]^p} \frac{dt}{t} = C \frac{\lambda^\theta}{\left[K\left(\frac{1}{\lambda}, f\right)\right]^{p-1}}$$

or

$$\left[\frac{\lambda^\theta}{\varphi(\lambda)}\right]^{p'} \leqq C\lambda^\theta K\left(\frac{1}{\lambda}, f\right)$$

and it follows at once the crucial condition:

$$\int_0^\infty \left(\frac{\lambda^\theta}{\varphi(\lambda)}\right)^{p'} \frac{d\lambda}{\lambda} < \infty.$$

Thus we have shown that f satisfies (2. 5).

We conclude by mentioning a sort of dual result.

Theorem 2. 3. *We have*

(2. 6) $$f \in (L_p(w_0), L_p(w_1))_{\theta, \infty}$$

if and only if

(2. 7) $$f \in L_p\left(w_0 \varphi\left(\frac{w_1}{w_0}\right)\right)$$ *for all non-decreasing* φ *such that*

$$\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t} < \infty.$$

Prof. i) Assume that (2. 7) holds true. We claim that then

(2.8) $$\sup \|f\|_{L_p\left(w_0 \varphi\left(\frac{w_1}{w_0}\right)\right)} \Bigg/ \left[\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t}\right]^{\frac{1}{p}} < \infty.$$

Indeed if this is not the case there is a sequence of non-decreasing functions φ_ν such that

$$\int_Z \left|w_0 \varphi_\nu\left(\frac{w_1}{w_0}\right) f\right|^p d\mu = 1, \quad \int_0^\infty \left(\frac{\varphi_\nu(t)}{t^\theta}\right)^p \frac{dt}{t} \leqq \frac{1}{\nu^2}.$$

If we set

$$(\varphi(t))^p = \sum_\nu (\varphi_\nu(t))^p$$

then φ is obviously non-decreasing too and

$$\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t} \leqq \sum_\nu \frac{1}{\nu^2} < \infty$$

but

$$\int\limits_Z \left| w_0 \varphi\left(\frac{w_1}{w_0}\right) f \right|^p d\mu = \sum_\nu 1 = \infty.$$

Thus we get a contradiction[1]). Upon taking $\varphi(t)=\min(1, t\, t_0)$ in (2. 8) we get

$$\int\limits_Z |\min(w_0, t_0 w_1) f|^p \, d\mu \leqq C^p t_0^{\theta p},$$

so by (2. 1) we get (2. 6).

ii) Conversely asume (2. 6) holds true. Then by (2. 1)

$$\int\limits_Z |\min(w_0, t w_1) f|^p \, d\mu \leqq C^p t^{\theta p}.$$

Multiply by $\left(\varphi\left(\frac{1}{t}\right)\right)^p$ and integrate! We get

$$\int\limits_Z \left[\int\limits_0^\infty \left[\varphi\left(\frac{1}{t}\right) \min(w_0, t w_1)\right]^p \frac{dt}{t}\right] |f|^p \, d\mu \leqq C^p \int\limits_0^\infty t^{\theta p} \left[\varphi\left(\frac{1}{t}\right)\right]^p \frac{dt}{t} < \infty.$$

But, since φ in non-decreasing,

$$\int\limits_0^\infty \left[\varphi\left(\frac{1}{t}\right) \min(w_0, t w_1)\right]^p \frac{dt}{t} \geqq \int\limits_{\frac{w_0}{2 w_1}}^{\frac{w_0}{w_1}} \left[\varphi\left(\frac{w_1}{w_0}\right) t w_1\right]^p \frac{dt}{t} \geqq C^p \left(\varphi\left(\frac{w_1}{w_0}\right)\right)^p w_0^p.$$

Thus we have established (2. 7).

Remark 2. 1. Theorem 2. 2 and 2. 3 should be compared to the known result ([see [7])

$$(L_p(w_0), L_p(w_1))_{\theta,p} = L_p(w_0^{1-\theta} w_1^\theta).$$

It would be inteiesting to have a characterization of $(L_p(w_0), L_p(w_1))_{\theta,q}$ which covers the remaining case $1 < q < \infty$, $q \neq p$ too.

3. Applications to Fourier integrals

We consider the space $\dot{W}_p^{s,q}$ of functions f on $(-\infty, \infty)$ such that

$$\left[\int\limits_0^\infty \left\| f(x+Nt) - \binom{N}{1} f(x+(N-1)t) + \ldots + (-1)^N f(x) \right\|_{L_p}^q \frac{dt}{t^{sq+1}}\right]^{1/q} < \infty$$

[1]) We owe the above argument to Nils-Olof Wallin.

where $1<p<\infty$, $1\leqq q\leqq\infty$, $0<s<N$. This definition is essentially independent of N. In view of [3], [5], [6] we have

$$\dot{W}_p^{s,q}=(L_p,\dot{H}_p^N)_{\frac{s}{N},q}$$

where $\dot{H}_p^N$ is the space of functions whose Nth order derivative is in L_p (with respect to Haar measure).

Now take $p=2$. By PARSEVAL's formula we then get

(3. 1) $$f\in\dot{W}_2^{s,q} \text{ if and only if } \hat{f}\in[L_2(1),L_2(|\xi|^N)]_{\frac{s}{N},q}$$

Thus we may apply the result of Section 2.

Let us write

$$L_p^{s,q}=[L_p(1),L_p(|\xi|^N)]_{\frac{s}{N},q}.$$

On applying theorem 2. 1 ($q=\infty$, $i=1$) we get: $\hat{f}\in L_p^{s,\infty}$ if and only if

(3. 2) $$\Big[\int\limits_{|\xi|\geqq 1/t}|\hat{f}(\xi)|^p\,d\xi\Big]^{1/p}\leqq Ct^s.$$

In view of the duality theorem of LIONS (see [4] or [5], chap. III) this is clearly related to [2], theorem II.

On applying theorem 2. 2 we get: $\hat{f}\in L_p^{s,1}$ if and only if

(3. 3) $$\int|\hat{f}(\xi)\varphi(|\xi|)|^p\,d\xi<\infty$$

for some *non-decreasing* φ such that

$$\int\limits_0^\infty\left(\frac{t^s}{\varphi(t)}\right)^{p'}\frac{dt}{t}<\infty.$$

In particular ($p=2$) we get by (3. 1): $f\in\dot{W}_2^{s,1}$ if and only if

(3. 4) $$\int|\hat{f}(\xi)\varphi(|\xi|)|^2\,d\xi<\infty$$

for some *non-decreasing* φ such that

$$\int\limits_0^\infty\left(\frac{t^s}{\varphi(t)}\right)^2\frac{dt}{t}<\infty.$$

If $s=\frac{1}{2}$ the condition on φ reads simply $\int\limits_0^\infty\frac{dt}{(\varphi(t))^2}<\infty$ and setting $\omega(t)=(\varphi(t))^2$ we get (0. 2). This is essentially [2], theorem III.

As in [2] similar results hold if $|\xi|^N$ is replaced by $(1+|\xi|)^N$.

It is also clear that we in the above fashion can treat the case of Fourier integrals in any number of variables.

References

[1] A. Beurling, On the spectral synthesis of bounded functions, *Acta Math.*, **81** (1949), 225—228.
[2] ——— Construction and analysis of some convolution algebras, *Ann. Inst. Fourier*, **14** (1964), 1—32.
[3] J. L. Lions, Théorèmes de traces et d'interpolation, *Ann. Scuola Normale Sup. Pisa*, **13** (1959), 389—403.
[4] ——— Sur les espaces d'interpolation; dualité, *Math. Scand.*, **9** (1961), 147—177.
[5] ——— and J. Peetre, Sur une classe d'espaces d'interpolation, *Inst. Hautes Etudes Sci. Publ. Math.*, **19** (1964), 5—68.
[6] J. Peetre, Espaces d'interpolation, généralisations, applications, *Rend. Sem. Mat. Fis. Milano*, **34** (1964), 133—161.
[7] ——— On an interpolation theorem of Foiaş and Lions, *Acta Sci. Math.*, **25** (1964), 255—261.
[8] ——— Applications de la théorie des espaces d'interpolation dans l'Analyse Harmonique, *Ricerche Mat.*, **15** (1966), 3—36.
[9] A. Zygmund, *Trigonometrical series* (Cambridge, 1958).

(Received March 5, 1966)