On multiplicative characters

By K. A. CORRADI and I. KÁTAI in Budapest

Let $f(n)$ be a multiplicative number-theoretical function, i. e.

$$f(mn)=f(m)f(n), \qquad (m, n)=1,$$

satisfying

(1) $$|f(n)| \leqq 1 \qquad (n=1, 2, \dots).$$

H. DELANGE [1] proved that for the fulfilment of the relation

(2) $$M(x) \overset{\text{def}}{=} \sum_{n \leqq x} f(n) = o(x)$$

a sufficient condition is given by (1) and

(3) $$\sum_p \frac{1-\operatorname{Re} f(p)}{p} = +\infty.$$

It is natural to ask for a condition, which turns out to be besides (1) sufficient for the fulfilment of the relation

(4) $$M(x; k, l) \overset{\text{def}}{=} \sum_{\substack{n \leqq x \\ n \equiv l \pmod k}} f(n) = o(x).$$

It is clear that (2) alone is a too weak condition for the fulfilment of (4). If for example $f(n)=\chi(n)$, where $\chi(n)$ stands for an arbitrary but fixed non-principal character mod k, then

$$M(x) = \sum_{n \leqq x} \chi(n) = O(x),$$

and on the other-hand for every l with $(l, k)=1$, $1 \leqq l \leqq k$,

$$M(x; k, l) = \sum_{\substack{n \leqq x \\ n \equiv l \pmod k}} \chi(n) = \chi(l) \left\{ \left[\frac{x-l}{k} \right] + 1 \right\} \neq o(x).$$

We will show in the sequel that the characters are exceptional in a certain sense.

Theorem 1. *Let $f(n)$ be an arbitrary but fixed multiplicative function satisfying (1). Let k and l be given natural numbers with $(k, l)=1$. Suppose that*

$$f(p)=\alpha, \quad \text{if} \quad p \equiv l \pmod k), \ p \text{ prime},$$

and suppose that the value of α *is different from all the values* $\chi(l)$ *taken by any* χ mod k. *Then for every* m *satisfying* $(m, k)=1$

$$\sum_{\substack{n\leqq x\\ n\equiv m (\mathrm{mod}\, k)}} f(n) = o(x) \tag{5}$$

holds.

Theorem 1 is a consequence of the following

Theorem 2. *Let* $g(n)$ *be an arbitrary but fixed multiplicative function satisfying* (1). *Suppose that for a pair* k, l *of coprime natural numbers*

$$g(p)=\beta, \quad \text{if} \quad p\equiv l\ (\mathrm{mod}\ k),\ p \text{ prime}$$

holds, where $\beta\neq 1$. *Then*

$$\sum_{n\leqq x} g(n) = o(x).$$

Deduction of Theorem 1 from Theorem 2. Let $g_\chi(n)=\chi(n)f(n)$, where $\chi(n)$ stands for an arbitrary character mod k. Then Theorem 2 applies by trivial arguments and gives the relation

$$\sum_{n\leqq x} \chi(n)f(n) = o(x).$$

From this, using

$$\sum_{\substack{n\leqq x\\ n\equiv m (\mathrm{mod}\, k)}} f(n) = \frac{1}{\varphi(k)}\sum_{\chi}\bar{\chi}(m)\sum_{n\leqq x}\chi(n)f(n),$$

the statement of Theorem 1 follows.

For the proof of Theorem 2 we need three lemmas. Before formulating the first of them we quote the following preliminaries.

Let $\mathcal{P}$ be an arbitrary infinite subset of the rational primes. Let the function $V_{\mathcal{P}}(n)$ be defined by

$$V_{\mathcal{P}}(n) \overset{\mathrm{def}}{=} \sum_{\substack{p\mid n\\ p\in\mathcal{P}}} 1. \tag{6}$$

$V_{\mathcal{P}}(n)$ is an additive function. Let the number A_x be defined by

$$A_x \overset{\mathrm{def}}{=} \sum_{\substack{p\leqq x\\ p\in\mathcal{P}}} \frac{1}{p}. \tag{7}$$

Then the following result of P. Turán [2] yields, which we state as

Lemma 1.

$$\sum_{n\leqq x} |V_{\mathcal{P}}(n)-A_x| = O(xA_x^{1/2}),$$

$$\sum_{n\leqq x} (V_{\mathcal{P}}(n)-A_x)^2 = O(xA_x).$$

Lemma 2. *For a coprime pair k, l of natural numbers*

$$\sum_{\substack{p \leqq x \\ p \equiv l (\mathrm{mod}\, k)}} \frac{1}{p} = \frac{1}{\varphi(k)} \log \log x + o(1).$$

This is an easy consequence of the prime-number theorem for arithmetical progressions.

Lemma 3. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Then in the notation of* (2) *and* (7)

$$M(x) A_x = \sum_{\substack{p \leqq x \\ p \in \mathcal{P}}} f(p) M\left(\frac{x}{p}\right) + O(x A_x^{1/2}). \tag{8}$$

For the proof, see [3].

Proof of Theorem 2. Let us define $\mathcal{P}$ as the set of rational primes satisfying

$$p \equiv l (\mathrm{mod}\, k). \tag{9}$$

For the sake of brevity we take

$$h \stackrel{\mathrm{def}}{=} \frac{1}{\varphi(k)} \log \log x. \tag{10}$$

Then using lemma 3 and the condition of Theorem 2 we have

$$M(x) h = \beta \sum_{\substack{p \leqq x \\ p \equiv l (\mathrm{mod}\, k)}} M\left(\frac{x}{p}\right) + O(x h^{1/2}). \tag{11}$$

We have to deal with two cases: 1) $|\beta| < 1$ and 2) $|\beta| = 1$. In the first case the statement of Theorem 2 can be deduced from (11) in a very simple way.

Take

$$\overline{\lim_{x \to \infty}} \frac{|M(x)|}{x} = \tau. \tag{12}$$

Then by (11)

$$|M(x)| h \leqq |\beta| (\tau + \varepsilon) x \sum_{\substack{p \leqq x \\ p \equiv l (\mathrm{mod}\, k)}} \frac{1}{p} + o(xh)$$

holds for every $\varepsilon > 0$ when $x \geqq x_0(\varepsilon)$. From this one can deduce by lemma 2 the inequality $|\tau| \leqq |\beta| \tau$.

In the case $|\beta| < 1$ this implies $\tau = 0$ i. e. the statement of the theorem is then true.

So it only remains to deal with the case $|\beta| = 1$. Let δ be an arbitrary but fixed number with $0 < \delta < \frac{1}{2}$. Using

$$\sum_{x^\delta \leqq p \leqq x} \left| M\left(\frac{x}{p}\right) \right| = O\left(x \sum_{x^\delta \leqq p \leqq x} \frac{1}{p}\right) = O(x)$$

and applying (11) with x/q instead of x, where $q \leqq x^{\delta}$ stands for a prime $\equiv l \pmod k$, we deduce by adding the results that

$$h \sum_{\substack{q \leqq x^{\delta} \\ q \equiv l \pmod k}}' M\left(\frac{x}{q}\right) = \beta \sum_{\substack{p \leqq x^{\delta},\, q \leqq x^{\delta} \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}). \tag{13}$$

Remark that by the deduction of (13) we have used the relation

$$\log\log\frac{x}{q} = \log\log x + O(1), \qquad q \leqq x^{\delta}.$$

From (13), by the modified form of (11), it follows that

$$M(x)h^2 = \beta^2 \sum_{\substack{p \leqq x^{\delta} \\ q \leqq x^{\delta} \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}). \tag{14}$$

The first term on the right-hand side of (14) can be described as

$$2\beta^2 \sum_{\substack{pq \leqq x^{\delta} \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + o(xh).$$

From (11), multiplying by h, and from (14) we obtain finally

$$h^2\left(\frac{1}{\beta} - \frac{1}{2\beta^2}\right) M(x) = h \sum_{\substack{p \leqq x^{\delta} \\ p \equiv l \pmod k}} M\left(\frac{x}{p}\right) - \sum_{\substack{pq \leqq x^{\delta} \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}). \tag{15}$$

Let now ε be an arbitrary but fixed positive number, and let us define the numbers n_ν by $n_\nu = (1+\varepsilon)^\nu$ $(\nu = 1, 2, \ldots)$. Then

$$\left| M\left(\frac{x}{n_\nu}\right) - M\left(\frac{x}{m}\right) \right| \leqq \left| \frac{x}{n_\nu} - \frac{x}{m} \right| + 1 \leqq \frac{\varepsilon x}{m} + 1$$

holds for $m \in [n_\nu, n_{\nu+1}]$. So by (15) we have

$$h^2\left(\frac{1}{\beta} - \frac{1}{2\beta^2}\right) M(x) = \sum_{\nu \leqq \frac{\delta \log x}{\log(1+\varepsilon)}} a_\nu M\left(\frac{x}{n_\nu}\right) + O(\varepsilon x h^2), \tag{16}$$

where a_ν is given by

$$a_\nu = h \sum_{\substack{p \in [n_\nu, n_{\nu+1}] \\ p \equiv l \pmod k}} 1 - \sum_{\substack{pq \in [n_\nu, n_{\nu+1}] \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} 1. \tag{17}$$

From the prime-number theorem for arithmetical progressions it follows that

$$h \sum_{\substack{p \in [n_\nu, n_{\nu+1}] \\ p \equiv l \pmod{k}}} = \frac{h\varepsilon n_\nu}{\varphi(k) \log n_\nu} (1+o(1)),$$

$$\sum_{\substack{pq \in [n_\nu, n_{\nu+1}] \\ p \leqq q \\ p \equiv q \equiv l \pmod{k}}} = \frac{1}{\varphi^2(k)} \cdot \frac{\varepsilon n_\nu}{\log n_\nu} \log\log n_\nu (1+o(1)).$$

So for $n_\nu \leqq x^{\delta/2}$ we have

$$a_\nu = \frac{1}{\varphi^2(k)} \frac{\varepsilon n_\nu}{\log n_\nu} \log \frac{\log x}{\log n_\nu} (1+o(1)). \tag{18}$$

Now as in the case $|\beta| < 1$ we have to show that $\tau = 0$. From (16) it follows by (18) that

$$h^2 \left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| |M(x)| \leqq \frac{x(\tau+\varepsilon)}{\varphi^2(k)} \varepsilon \sum_{n_\nu \leqq x^\delta} \frac{\log\log x - \log\log n_\nu}{\log n_\nu} + o(\varepsilon x h^2). \tag{19}$$

From (19) one has the inequality

$$h^2 \left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| |M(x)| \leqq \frac{1}{2} x(\tau+\varepsilon) \frac{\varepsilon}{\log(1+\varepsilon)} h^2 + o(x\varepsilon h^2). \tag{20}$$

Letting first $x \to \infty$ then $\varepsilon \to 0$ it results

$$\left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| \tau \leqq \frac{\tau}{2}.$$

Hence $\tau = 0$ or $\left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| \leqq \frac{1}{2}$. The second possibility can not occur in our case. For it implies $|\beta| \geqq 1$ and it holds in the case $|\beta| = 1$ for $\beta = 1$ only. But $\beta = 1$ is impossible by the assumption of the theorem. Thus $\tau = 0$ and Theorem 2 is proved.

A similar argument leads to the following

Theorem 3. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Suppose that $f(p) = \alpha$ if $p \equiv l \pmod{k}$, p prime, holds for some k, l with $(k, l) = 1$, where $|\alpha| < 1$ or $\frac{\arg \alpha}{2\pi}$ is an irrational number. Then for every pair k^*, l^* of natural numbers*

$$\sum_{\substack{n \leqq x \\ n \equiv l^* \pmod{k^*}}} f(n) = o(x).$$

We note for the proof, that it is sufficient to deal with the case $(l^*, k^*) = 1$ only. In this case it is enough to show the fulfilment of

$$\sum_{n \leqq x} g_\chi(n) = o(x),$$

where $g_\chi(n) = f(n)\chi(n)$ and $\chi(n)$ stands for an arbitrary but fixed character mod k^*.

The proof of Theorem 2 applies in this case too, the only difference is in the choice of the set $\mathcal{P}$ of Theorem 2, which requires but obvious modifications. We omit the details.

As an immediate consequence of Theorem 2 we mention the following

Theorem 4. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Suppose that for a given natural number k and for every l coprime to k*

$$f(p)=\alpha_l \quad \textit{if} \quad p\equiv l \pmod k, \ p \ \textit{prime}.$$

Suppose further that there exists for every character $\chi(n)$ mod k *at least one prime p with*

$$\chi(p)\neq f(p).$$

Then for all l with $1\leqq l\leqq k$, $(k,l)=1$, we have

$$\sum_{\substack{n\leqq x\\ n\equiv l \pmod k}} f(n) = o(x).$$

Observe that Theorem 3 has some consequences for the distribution of the fractional parts of the values taken by an additive function. Let $h(n)$ be an additive function i. e. $h(mn)=h(m)+h(n)$ for $(m, n)=1$. Suppose that

$$h(p)=\alpha \quad \text{if} \quad p\equiv l \pmod k, \ p \ \text{prime}$$

holds for a given coprime pair k, l of natural numbers, where α denotes an arbitrary irrational number.

Under these conditions we have the following

Theorem 5. *The values of $\{h(n)\}$ are uniformly distributed in every arithmetical progression.* (Here { } stands for the fractional part.)

The proof of Theorem 5 can be obtained by applying Theorem 3 to the functions

$$f_t(n) = e^{2\pi i t h(n)}$$

and using WEYL's theorem [4] concerning uniform distribution.

References

[1] H. DELANGE, Un théorème sur les fonctions arithmétiques multiplicatives et ses applications, *Ann. Ec. Norm. Sup.*, (3) **78** (1961), 1—29.

[2] P. TURÁN, Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan, *J. London Math. Soc.*, **11** (1963), 125—133.

[3] K. A. CORRADI and I. KÁTAI, On the theory of multiplicative functions, *Annales Univ. Sci. Budapest, sect. math.*, **9** (1966), 147—155.

[4] H. WEYL, Über die Gleichverteilung mod 1, *Math. Ann.*, **77** (1916), 313—352.

(Received July 9, 1966)