An example in the theory of Fourier series

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A. PLESSNER proved in [1] that if a trigonometric series converges on a set E with $mE>0$, then its conjugate series converges almost everywhere on E. This fact was proved independently by J. MARCINKIEWICZ and A. ZYGMUND [2] too.

In the present note we are going to prove the following

Theorem. *There exists a sequence $\{\alpha_n\}$ of non-negative numbers with the following properties. The series*

(1) $$\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n\cos nx$$

is the Fourier series of an $f(x)\geqq 0$, $f(x)\in L^q[-\pi,\pi]$ for every $q>0$. The series (1) diverges unboundedly on an everywhere dense set of second category in $[-\pi,\pi]$. There exists an infinite sequence of natural numbers with

(2) $$s_{n_k}(x)=\frac{\alpha_0}{2}+\sum_{\nu=1}^{n_k}\alpha_\nu\cos\nu x\geqq 0, \qquad (x\in[-\pi,\pi];\quad k=1,2,\ldots).$$

Finally the series

(3) $$\sum_{n=1}^{\infty}\alpha_n\sin nx$$

converges uniformly in $[-\pi,\pi]$, and so proves to be the Fourier series of a continuous function.

Remark. By the quoted result of PLESSNER the series (1) converges almost everywhere on $[-\pi,\pi]$.

For the proof of the Theorem we need three lemmas.

Lemma 1. *Suppose that all partial sums of the series*

(4) $$\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n\cos nx$$

are non-negative for every $x\in[-\pi,\pi]$. Then the same is true for the series

(5) $$\frac{b_0^2}{2}+\sum_{n=1}^{\infty}b_n^2\cos nx.$$

Proof. Let us denote the partial sums of (4) by

$$u_n(x) = \frac{b_0}{2} + \sum_{\nu=1}^{n} b_\nu \cos \nu x, \qquad (n=0, 1, 2, \ldots).$$

So we have

$$\frac{b_0^2}{2} + \sum_{\nu=1}^{n} b_\nu^2 \cos \nu x = \frac{1}{\pi} \int_{-\pi}^{\pi} u_n(t) u_n(x-t)\, dt$$

which, by the supposed non-negativity of $u_n(x)$, proves the statement of the lemma.

Lemma 2. *Let $\{c_n\}$ be a decreasing sequence of positive numbers satisfying $c_n = 0\left(\frac{1}{n}\right)$. Then*

$$\left| \sum_{\nu=1}^{n} c_\nu \sin \nu x \right| < K \qquad (n=1, 2, \ldots).$$

This lemma represents a well-known result (see [3], Vol. I. pp. 182—183). The third thing we need is a theorem of P. Turán, which we formulate as

Lemma 3. *All partial sums of the series*

$$1 + \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \cos nx \tag{6}$$

are non-negative for every $x \in [-\pi, \pi]$.

Before going to prove the Theorem, we mention that the coefficients of (6) form a decreasing sequence satisfying

$$c_1 n^{-\frac{1}{2}} < (-1)^n \binom{-\frac{1}{2}}{n} < c_2 n^{-\frac{1}{2}} \tag{7}$$

with suitable $c_1, c_2 > 0$.

Let now $\{\lambda_r\}$ be an arbitrary but fixed sequence of positive numbers with

$$\sum_{r=1}^{\infty} \frac{1}{\lambda_r} < +\infty.$$

We introduce the notation

$$a_0 = 4, \quad a_n = \binom{-\frac{1}{2}}{n}^2 \qquad (n = 1, 2, \ldots).$$

Then for every natural number u we have by (7)

$$\frac{a_0}{2} + \sum_{v=1}^{V} a_{uv} > B \tag{8}$$

for any $B > 0$ if $V = V(B)$ is large enough.

As the next step we define two sequences $\{l_r\}$ and $\{m_r\}$ of natural numbers by induction on r.

Let $l_1=1$ and let m_1 be the least natural number satisfying the condition

$$\frac{a_0}{2}+\sum_{\nu=1}^{m_1} a_\nu > \frac{1}{\lambda_1}.$$

The choice of m_1 is always possible by (8).

Suppose now that the numbers l_u and m_u are already defined for $1<u<r-1$. Determine l_r and m_r by the conditions

1. l_r is the least natural number which is a multiple of l_{r-1} and satisfying

$$l_r > l_{r-1} m_{r-1}.$$

2. If l_r is chosen as mentioned, m_r is the least natural number satisfying

$$\frac{a_0}{2}+\sum_{\nu=1}^{m_r} a_{\nu l_r} > \frac{1}{\lambda_r}.$$

The choice of m_r is always possible by (8). Thus the sequences $\{l_r\}$ and $\{m_r\}$ are defined for every value of $r \geqq 1$.

Let now the trigonometric polynomials $U_r(x)$ and $V_r(x)$ be defined by

$$(9)\qquad U_r(x)=\frac{a_0}{2}+\sum_{\nu=1}^{m_r} a_{\nu l_r}\cos \nu l_r x,\quad V_r(x)=\sum_{\nu=1}^{m_r} a_{\nu l_r}\sin \nu l_r x \qquad (r=1,2,\ldots).$$

We shall show that

$$(10)\qquad \sum_{r=1}^{\infty}\frac{1}{\lambda_r}U_r(x)$$

and

$$(11)\qquad \sum_{r=1}^{\infty}\frac{1}{\lambda_r}V_r(x)$$

are conjugate trigonometric series possessing the properties required in the Theorem.

First of all, observe that the series

$$(12)\qquad \sum_{n=1}^{\infty} a_n \sin nx$$

has monotonically decreasing coefficients satisfying $a_n=O\left(\frac{1}{n}\right)$ by (7) and (8).

Thus, using Lemma 2, we get that

$$(13)\qquad \left|\sum_{\nu=1}^{n} a_\nu \sin \nu x\right| < K \qquad (n=1,2,\ldots)$$

holds for every $x\in[-\pi,\pi]$. On the other hand, we have

$$(14)\qquad V_r(x)=\frac{1}{l_r}\sum_{t=0}^{l_r-1}\left\{\sum_{\nu=1}^{m_r l_r} a_\nu \sin \nu\left(x+\frac{2\pi t}{l_r}\right)\right\}.$$

So one can deduce from (14), using (13), that

$$(15)\qquad |V_r(x)|<K \qquad (x\in[-\pi,\pi];\ r=1,2,\ldots).$$

holds. (15) means that the series (11) converges uniformly in $[-\pi, \pi]$. Thus

(16) $$\tilde{f}(x) \overset{\text{def}}{=} \sum_{r=1}^{\infty} \frac{1}{\lambda_r} V_r(x)$$

is a continuous function. By the choice of the sequences $\{l_r\}$ and $\{m_r\}$, $V_r(x)$ and $V_s(x)$ do not contain common sines if $r \neq s$.

By the uniform convergence of (16) and the remark done before we have

(17) $$\alpha_n = \begin{cases} \frac{1}{\lambda_r} a_{\nu l_r} & \text{if} \quad n = \nu l_r, \quad 1 \leqq \nu \leqq m_r \\ 0 & \text{otherwise,} \end{cases}$$

where α_n denotes the nth Fourier sine coefficient of $\tilde{f}(x)$. By (17), using the definition of the sequences $\{a_n\}$ and $\{\lambda_r\}$ we get the inequality

(18) $$\alpha_n \geqq 0 \qquad (n = 1, 2, \ldots).$$

Now (18) means by a theorem of PALEY [5], that the Fourier series of $\tilde{f}(x)$ converges uniformly.

A representation similar to (14) shows that

$$U_r(x) \geqq 0 \qquad (x \in [-\pi, \pi];\ r = 1, 2, \ldots)$$

holds. Indeed, by Lemma 3 and Lemma 1 all the partial sums of the series

(19) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

are non-negative for every $x \in [-\pi, \pi]$. Using this fact we get

(20) $$\sum_{r=1}^{\infty} \frac{1}{\lambda_r} \int_{-\pi}^{\pi} U_r(x)\, dx < +\infty$$

which means by the theorem of Beppo Levi, that the series (10) converges a. e. on $[-\pi, \pi]$ to an $f(x) \geqq 0$, $f(x) \in L[-\pi, \pi]$.

A similar argument shows that

(21) $$f(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos nx.$$

$f(x)$ and $\tilde{f}(x)$ being given by conjugate Fourier-series and since $\tilde{f}(x) \in L^q[-\pi, \pi]$ for every $q > 0$, it follows by known arguments that $f(x) \in L^q[-\pi, \pi]$ holds for every $q > 0$.

Now we note that the series (21) diverges unboundedly on an everywhere dense set of second category in $[-\pi, \pi]$. In fact

$$\frac{1}{\lambda_r} U_r\left(2\pi \frac{l}{l_s}\right) \geqq 1, \qquad 0 \leqq l < l_s - 1, \quad r \geqq s$$

holds by $l_s | l_r$ if $r \geqq s$ and by the choice of the numbers $\{l_r\}$ and $\{m_r\}$. From this, noting that the numbers $2\pi \frac{l}{l_s}$ $(s=1, 2, \ldots)$ lie everywhere dense in $[0, 2\pi]$, the assertion follows. We quote the result that if a series of continuous functions diverges unboundedly on an everywhere dense set in $[-\pi, \pi]$, then the set of points where the series diverges unboundedly is of the second category in $[-\pi, \pi]$. For the proof see for example [6].

We conclude by remarking that for the choice $n_k = l_k m_k$ the non-negativity of the partial sums required in the theorem follows without difficulty.

References

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