A remark on the theory of multiplicative functions

By K. A. CORRADI in Budapest

In [2] E. M. WRIGHT presents a proof of the prime-number theorem, which uses elementary methods and depends on ideas introduced by A. SELBERG in the theory of numbers. His method used there enables one to prove more general results of the same character concerning multiplicative functions. We call in the sequel a function $f(n)$, defined on the domain of the natural numbers, *multiplicative* if for coprime integers m and n the relation

$$f(mn)=f(m)f(n)$$

holds. In this note we prove a theorem concerning multiplicative functions of that kind. It can be stated as follows.

Theorem. *Let $f(n)$ be a multiplicative function, which takes the three values $0, 1, -1$ only. Let k be a positive integer. Suppose that there exists a natural number l, for which $l\leqq k$ and $(l, k)=1$, and the relation*

$$\sum_{\substack{n\leqq x\\ n\equiv l\,(\mathrm{mod}\,k)}} \frac{f(n)}{n}=O(1)$$

is satisfied. Then for all u with $1\leqq u\leqq k$, $(u, k)=1$, the relation

$$F_u(x)\overset{\mathrm{def}}{=}\sum_{\substack{n\leqq x\\ n\equiv u\,(\mathrm{mod}\,k)}} f(n)=o(x)$$

holds.

We mention, that in the case $k=1$, $l=1$, when

$$f(n)\equiv\mu(n),$$

where $\mu(n)$ stands for MOEBIUS's function, our result presents the prime number theorem. For the detailed elementary deduction of the prime number theorem from $\sum_{n\leqq x} u(n)=o(x)$, and for an elementary proof of the mentioned relation (and thus of the prime number theorem) see [3].

The proof of the theorem consists of two different parts. In the first part we prove the inequality

$$(1)\qquad |F_u(x)|\log^2 x \leqq \frac{2}{\varphi(k)} \sum_{\substack{v\leqq k\\ (v,k)=1}} \int_1^x \left|F_v\left(\frac{x}{t}\right)\right| \log t\, dt + O(x\log x),$$

where $\varphi(k)$ denotes EULER's function, and in the second one we deduce from (1) the statement of the theorem.

For the sake of completeness we give the proof in full details. The method used by us shows a great formal similarity to that of [2]. The difference between the two methods lies primarily in the fact, that we make use, besides the original formulas of SELBERG, of some further formulas closely related to them. In the paper we make free use of the terminology applied in [2]. Before concluding these preliminary remarks, we observe that the result of the paper does not seem to be obtainable by analytical methods, thus in this case elementary methods seem to go further than those of the theory of functions.

Proof of the first part of the theorem

We shall need the following lemmas:

Lemma 1. *Let $c_1, c_2, \ldots$ be a sequence of numbers,*

$$C(t) = \sum_{n\leqq t} c_n \qquad (1\leqq t<\infty),$$

and $f(t)$ a function of t. Then

$$\sum_{n\leqq x} c_n f(n) = \sum_{n\leqq x-1} C(n)\{f(n)-f(n+1)\} + C(x)f([x]).$$

If, in addition, $c_j=0$ for $j<n_1$ and $f(t)$ has a continuous derivative for $t\geqq n_1$, then

$$\sum_{n\leqq x} c_n f(n) = C(x)f(x) - \int_{n_1}^x C(t)f'(t)\,dt.$$

For the proof of the lemma see [1], theorem 421, p. 346.

Lemma 2. (SELBERG's formula.) *Let k be a positive integer. Then if $1\leqq u\leqq k$, $(u,k)=1$, we have*

$$(2)\qquad \sum_{\substack{n\leqq x\\ n\equiv u\ (\mathrm{mod}\ k)}} \Lambda(n)\log n + \sum_{\substack{mn\leqq x\\ mn\equiv u\ (\mathrm{mod}\ k)}} \Lambda(m)\Lambda(n) = \frac{2}{\varphi(k)} x\log x + O(x).$$

Here $\varphi(k)$ stands for EULER's function. For the proof see [4].

Lemma 3. *Let $f(n)$ be any multiplicative function satisfying the condition*

$$|f(n)|\leqq 1$$

for all values of the positive integer n. Then

$$F_u(x)\log x-\sum_{n\leq x} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)=O(x), \tag{3}$$

where $\Lambda(n)$ *denotes* VON MANGOLDT'*s function, and* n^{-1} *stands for the number* m *determined by the conditions*

$$mn\equiv 1 \pmod k), \qquad 1\leq m\leq k.$$

Proof. We start with the obvious formula

$$\sum_{\substack{n\leq x\\ n\equiv u \pmod k}} f(n)\log\frac{x}{n}=O(x).$$

Hence

$$F_u(x)\log x-\sum_{\substack{n\leq x\\ n\equiv u \pmod k}} f(n)\log n=F_u(x)\log x-\sum_{p\leq x} f(p)\log p\, F_{up^{-1}}\left(\frac{x}{p}\right)+O(x)=O(x).$$

Now considering that

$$\Lambda(p)=\log p \quad\text{and}\quad \sum_{n\leq x,\, n\neq p} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)=O(x),$$

we get at once the statement of the lemma.

After these preliminaries we perform the first part of the proof, i. e. the proof of the inequality (1).

If we replace n by m and x by x/n in (3), we have

$$F_u\left(\frac{x}{n}\right)\log\frac{x}{n}-\sum_{m\leq\frac{x}{n}} f(m)\Lambda(m)F_{um^{-1}}\left(\frac{x}{mn}\right)=O(x).$$

Hence

$$\left\{F_u(x)\log x-\sum_{n\leq x} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)\right\}\log x+$$

$$+\sum_{n\leq x} f(n)\Lambda(n)\left\{F_{un^{-1}}\left(\frac{x}{n}\right)\log\frac{x}{n}-\sum_{m\leq\frac{x}{n}} f(m)\Lambda(m)F_{un^{-1}n^{-1}}\left(\frac{x}{mn}\right)\right\}=$$

$$=O(x\log x)+O\left(x\sum_{n\leq x}\frac{\Lambda(n)}{n}\right)=O(x\log x),$$

that is

$$F_u(x)\log^2 x=\sum_{n\leq x} f(n)\,\Lambda(n)\log n F_{un^{-1}}\left(\frac{x}{n}\right)+$$

$$+\sum_{mn\leq x} f(m)f(n)\Lambda(m)\Lambda(n)F_{um^{-1}n^{-1}}\left(\frac{x}{mn}\right)+O(x\log x),$$

whence

$$|F_u(x)|\log^2 x \leqq \sum_{\substack{1\leqq v\leqq k\\(v,k)=1}} \sum_{n\leqq x} a_n^{(v)} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| + O(x\log x), \tag{4}$$

where

$$a_n^{(v)} = \begin{cases} \Lambda(n)\log n + \sum_{hk=n} \Lambda(h)\Lambda(k), & \text{if} \quad n\equiv v (\text{mod } k),\\ 0 \quad \text{otherwise,} \end{cases}$$

and

$$\sum_{n\leqq x} a_n^{(v)} = \frac{2}{\varphi(k)} x\log x + O(x)$$

by (2).

We now replace the inner sum on the right-hand side of (4) by an integral. To do so, we shall prove that

$$\sum_{n\leqq x} a_n^{(v)} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| = \frac{2}{\varphi(k)}\int_1^x \left|F_{uv^{-1}}\left(\frac{x}{t}\right)\right| \log t\, dt + O(x\log x). \tag{5}$$

Noticing the fact, that if v runs through a restricted system of residues mod k, then the same is true of uv^{-1} for an arbitrary u with $(u, k)=1$, which means that (4) and (5) together will conclude the proof of (1).

We remark that if $t>t'\geqq 0$

$$||F_{uv^{-1}}(t)|-|F_{uv^{-1}}(t')|| \leqq |F_{uv^{-1}}(t)-F_{uv^{-1}}(t')| =$$

$$= |(F_{uv^{-1}}(t)+G_{uv^{-1}}(t))-(F_{uv^{-1}}(t')+G_{uv^{-1}}(t'))-G_{uv^{-1}}(t)+G_{uv^{-1}}(t')| \leqq$$

$$\leqq H_{uv^{-1}}(t)-H_{uv^{-1}}(t'),$$

where

$$H_{uv^{-1}}(t) \overset{\text{def}}{=} F_{uv^{-1}}(t)+2G_{uv^{-1}}(t), \qquad G_{uv^{-1}}(t) \overset{\text{def}}{=} \sum_{\substack{n\leqq t\\ n\equiv uv^{-1}}} 1,$$

and that $H_{uv^{-1}}(t)$ is a steadily increasing function of t, $H_{uv^{-1}}(t)=O(t)$. Using lemma 1, we obtain that

$$\sum_{n\leqq x-1} n\left\{H_{uv^{-1}}\left(\frac{x}{n}\right)-H_{uv^{-1}}\left(\frac{x}{n+1}\right)\right\} = \sum_{n\leqq x} H_{uv^{-1}}\left(\frac{x}{n}\right)-[x]H_{uv^{-1}}\left(\frac{x}{[x]}\right) =$$

$$= O\left(x\sum_{n\leqq x}\frac{1}{n}\right) = O(x\log x). \tag{6}$$

Now we prove (5) in two steps. First if we put

$$c_1=0, \quad c_n = a_n^{(v)} - \frac{2}{\varphi(k)}\int_{n-1}^{n} \log t\, dt, \qquad f(n) = \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right|$$

in the lemma 1, we have

$$C(x) = \sum_{n \leqq x} a_n^{(v)} - \frac{2}{\varphi(k)} \int_1^{[x]} \log t\, dt = O(x)$$

and

$$\sum_{n \leqq x} a_n^{(v)} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| - \frac{2}{\varphi(k)} \sum_{2 \leqq n \leqq x} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt =$$

$$(7) \qquad = \sum_{n \leqq x-1} C(n) \left\{\left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| - \left|F_{uv^{-1}}\left(\frac{x}{n+1}\right)\right|\right\} + C(x) F_{uv^{-1}}\left(\frac{x}{[x]}\right) =$$

$$= O\left(\sum_{n \leqq x-1} n \left\{H_{uv^{-1}}\left(\frac{x}{n}\right) - H_{uv^{-1}}\left(\frac{x}{n+1}\right)\right\}\right) + O(x) = O(x \lg x)$$

by (6).

Next

$$\left|\left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt - \int_{n-1}^{n} \left|F_{uv^{-1}}\left(\frac{x}{t}\right)\right| \log t\, dt\right| \leqq$$

$$\leqq \int_{n-1}^{n} \left|\left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| - \left|F_{uv^{-1}}\left(\frac{x}{t}\right)\right|\right| \log t\, dt \leqq$$

$$\leqq \int_{n-1}^{n} \left\{H_{uv^{-1}}\left(\frac{x}{t}\right) - H_{uv^{-1}}\left(\frac{x}{n}\right)\right\} \log t\, dt \leqq (n-1)\left\{H_{uv^{-1}}\left(\frac{x}{n-1}\right) - H_{uv^{-1}}\left(\frac{x}{n}\right)\right\}.$$

Hence

$$\sum_{2 \leqq n \leqq x} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt - \int_{1}^{x} \left|F_{uv^{-1}}\left(\frac{x}{t}\right)\right| \log t\, dt =$$

$$(8) \qquad = O\left(\sum_{n \leqq x-1} n \left\{H_{uv^{-1}}\left(\frac{x}{n}\right) - H_{uv^{-1}}\left(\frac{x}{n+1}\right)\right\}\right) + O(x \log x) = O(x \log x).$$

Combining (7) and (8), we get (5).

Proof of the second part of the theorem

First we give our inequality (1) another form. We introduce the functions

$$V_v(\xi)=e^{-\xi}F_v(e^{\xi}),\qquad 1\leqq v\leqq k,\ (v,k)=1.$$

If we write $x=e^{\xi}$, $t=xe^{-\eta}$, we have

$$\int_1^x\left|F_v\left(\frac{x}{t}\right)\right|\log t\,dt = x\int_0^{\xi}|V_v(\eta)|\,(\xi-\eta)\,d\eta = x\int_0^{\xi}|V_v(\eta)|\int_{\eta}^{\xi}d\zeta\,d\eta = x\int_0^{\xi}\int_0^{\zeta}|V_v(\eta)|\,d\eta\,d\zeta$$

by interchanging the order of integration. Then our inequality (1) becomes

(9) $$\xi^2|V_v(\xi)|\leqq\frac{2}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\int_0^{\xi}\int_0^{\zeta}|V_v(\eta)|\,d\eta\,d\zeta+O(\xi).$$

The functions $V_v(\xi)$ are bounded as $\xi\to\infty$. Hence we may write

$$\alpha_v=\overline{\lim_{\xi\to\infty}}\,|V_v(\xi)|,\quad \beta_v=\overline{\lim_{\xi\to\infty}}\,\frac{1}{\xi}\int_0^{\xi}|V_v(\eta)|\,d\eta,$$

since both these upper limits exist. Clearly

(10) $$|V_v(\xi)|\leqq\alpha_v+o(1),$$

and

$$\int_0^{\xi}|V_v(\eta)|\,d\eta\leqq\beta_v\xi+o(\xi).$$

Using this in (9), we get

$$\xi^2|V_u(\xi)|\leqq\frac{2}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\int_0^{\xi}\{\beta_v\zeta+o(\xi)\}\,d\zeta+O(\xi)=\xi^2\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v+o(\xi^2),$$

and from this

$$|V_u(\xi)|\leqq\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v+o(1).$$

Hence

(11) $$\alpha_u\leqq\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v.$$

In the sequel, let

(12) $$\alpha\overset{\text{def}}{=}\max_{\substack{1\leqq u\leqq k\\(u,k)=1}}\alpha_u.$$

To the completion of the proof it is enough to show that $\alpha=0$. We suppose that $\alpha>0$ and prove that this leads to a contradiction. For all v in question

$$\beta_v \leqq \alpha_v$$

holds by trivial arguments. So (11) gives that

$$\alpha \leqq \frac{1}{\varphi(k)} \sum_{\substack{v\leqq k\\(v,k)=1}} \beta_v \leqq \frac{1}{\varphi(k)} \sum_{\substack{v\leqq k\\(v,k)=1}} \alpha_v \leqq \alpha \tag{13}$$

and this can only hold if for all values of v

$$\beta_v=\alpha_v=\alpha$$

is fulfilled. Now from the assumption $\alpha>0$ we shall derive that

$$\beta_l<\alpha_l=\alpha$$

for the index l occurring in the theorem, and this will give a contradiction in the inequality (13).

For the proof we require two further lemmas.

Lemma 4. *Let the meaning of l be that of the theorem. Then there is a fixed positive number A, such that for every positive ξ_1, $\xi_2\geqq\xi_1$ we have*

$$\left|\int_{\xi_1}^{\xi_2} V_l(\eta)\,d\eta\right|<A.$$

Proof. If we put $x=e^{\xi}$, $t=e^{\eta}$, we have

$$\int_0^{\xi} V_l(\eta)\,d\eta=\int_1^x \frac{F_l(t)}{t^2}\,dt=\sum_{\substack{n\leqq x\\ n\equiv l \pmod k}} \frac{f(n)}{n}-\frac{F_l(x)}{x}=O(1)+O(1)=O(1)$$

using the condition for $f(n)$ required in the theorem, and applying lemma 1 with

$$c_n=\begin{cases} f(n), & \text{if } \ n\equiv l \pmod k,\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad f(t)=\frac{1}{t}.$$

Hence we get for the numbers ξ_1 and ξ_2 that

$$\left|\int_{\xi_1}^{\xi_2} V_l(\eta)\,d\eta\right|=\left|\int_0^{\xi_2} V_l(\eta)\,d\eta-\int_0^{\xi_1} V_l(\eta)\,d\eta\right|\leqq$$

$$\leqq\left|\int_0^{\xi_2} V_l(\eta)\,d\eta\right|+\left|\int_0^{\xi_1} V_l(\eta)\,d\eta\right|=O(1)+O(1)=O(1),$$

and this gives the statement of the lemma.

Lemma 5. *If* $\eta_0>0$, *and* $V_l(\eta_0)=0$, *then*

$$\int_0^\alpha |V_l(\eta_0+\tau)|\,d\tau \leqq \frac{2}{k}\alpha^2+O\left(\frac{1}{\eta_0}\right),$$

where the meaning of α *is the same as in* (12).

Proof. We start with a simple remark. If we put $T_u(x)$ for

$$\sum_{\substack{n\leqq x\\ n\equiv u\,(\mathrm{mod}\,k)}} 1$$

then as one can see it without difficulty these functions satisfy the relation

$$T_l(x)\log x+\sum_{n\leqq x}\Lambda(n)T_{ln^{-1}}\left(\frac{x}{n}\right)=\frac{2}{k}x\log x+O(x).$$

Combining this with (3) we have

$$\text{(14)}\qquad \{T_l(x)+F_l(x)\}\log x+\sum_{n\leqq x}\Lambda(n)Q_{ln^{-1}}\left(\frac{x}{n}\right)=\frac{2}{k}x\log x+O(x),$$

where $Q_{ln^{-1}}(y)$ stands for

$$\sum_{\substack{m\leqq y\\ m\equiv ln^{-1}\,(\mathrm{mod}\,k)}}\{1-f(n)f(m)\}.$$

If we take into account that the function $T_l(x)+F_l(x)$ steadily increases, and that for $Q_{ln^{-1}}(y)$

$$Q_{ln^{-1}}(y)\geqq 0$$

holds and this function has the same monotonity property as the function mentioned before, we get that for any positive x_0 and $x\geqq x_0$

$$0\leqq\{T_l(l)+F_l(x)\}\log x-\{T_l(x_0)+F_l(x_0)\}\log x_0\leqq\frac{2}{k}\{x\log x-x_0\log x_0\}+O(x).$$

From this we deduce that

$$\text{(15)}\qquad |F_l(x)\log x-F(x_0)\log x_0|\leqq\frac{1}{k}\{x\log x-x_0\log x_0\}+O(x)$$

by virtue of the trivial relation

$$T_l(x)=\frac{x}{k}+O(1).$$

We put $x=e^{\eta_0+\tau}$, $x_0=e^{\eta_0}$, so that $F_l(x_0)=0$. We have, since $0\leqq\tau\leqq\alpha$

$$|V_l(\eta_0+\tau)|\leqq\frac{1}{k}\left(1-\left(\frac{\eta_0}{\eta_0+\tau}\right)e^{-\tau}\right)+O\left(\frac{1}{\eta_0}\right)=$$

$$=\frac{1}{k}(1-e^{-\tau})+O\left(\frac{1}{\eta_0}\right)\leqq\frac{1}{k}\tau+O\left(\frac{1}{\eta_0}\right)$$

by (15), and so

$$\int_0^\alpha |V_l(\eta_0+\tau)|\,d\tau \leqq \frac{1}{k}\int_0^\alpha \tau\,d\tau + O\left(\frac{1}{\eta_0}\right) = \frac{1}{2k}\alpha^2 + O\left(\frac{1}{\eta_0}\right),$$

which gives the statement of the lemma.

We now write

$$\delta = \frac{2Ak+(4k-1)\alpha^2}{2k\alpha} > \alpha,$$

take any positive number ζ and consider the behaviour of $V_l(\eta)$ in the interval $\zeta \leqq \eta \leqq \zeta+\delta-\alpha$. By the definition of $V_l(\eta)$, this function can change sign in the interval mentioned above only in the case, if there is an η_0 lying in $(\zeta, \zeta+\delta-\alpha)$ for which $V_l(\eta_0)=0$, owing to the fact that $f(n)$ takes the three values $0, 1, -1$ only. Hence in our interval, either $V_l(\eta_0)=0$ for some η_0 or $V_l(\eta)$ does not change sign at all. In the first case, we use (10) and lemma 5, and have

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta = \int_\zeta^{\eta_0} + \int_{\eta_0}^{\eta_0+\alpha} + \int_{\eta_0+\alpha}^{\zeta+\delta} |V_l(\eta)|\,d\eta \leqq$$

$$\leqq \alpha(\eta_0-\zeta) + \frac{1}{2k}\alpha^2 + \alpha(\zeta+\delta-\eta_0-\alpha) + o(1) =$$

$$= \alpha\left(\delta - \left(1-\frac{1}{2k}\right)\alpha\right) + o(1) = \alpha'\delta + o(1)$$

for large ζ, where $\alpha' = \alpha\left(1-\left(1-\frac{1}{2k}\right)\frac{\alpha}{\delta}\right) < \alpha$. In the second one we have

$$\int_\zeta^{\zeta+\delta-\alpha} |V_l(\eta)|\,d\eta = \left|\int_\zeta^{\zeta+\delta-\alpha} V_l(\eta)\,d\eta\right| < A$$

by lemma 4. Hence

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta = \int_\zeta^{\zeta+\delta-\alpha} + \int_{\zeta+\delta-\alpha}^{\zeta+\delta} |V_l(\eta)|\,d\eta < A+\alpha^2+o(1) = \alpha''\delta + o(1),$$

where

$$\alpha'' = \frac{A+\alpha^2}{\delta} = \alpha\left(\frac{2kA+2k\alpha^2}{2kA+(4k-1)\alpha^2}\right) = \alpha\left(1-\left(1-\frac{1}{2k}\right)\frac{\alpha}{\delta}\right) = \alpha'.$$

Thus we have always

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta \leqq \alpha'\delta + o(1)$$

where $o(1)\to\infty$ as $\zeta\to\infty$. If $M=\left[\frac{\xi}{\delta}\right]$,

$$\int_0^{\xi}|V_l(\eta)|\,d\eta=\sum_{m=0}^{M-1}\int_{m\delta}^{(m+1)\delta}|V_l(\eta)|\,d\eta+\int_{M\delta}^{\xi}|V_l(\eta)|\,d\eta\leqq$$

$$\leqq\alpha' M\delta+o(M)+O(1)=\alpha'\xi+o(\xi).$$

Hence

$$\beta_l=\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\int_0^{\xi}|V_l(\eta)|\,d\eta\leqq\alpha'<\alpha,$$

and this inequality gives the contradiction as desired. So our theorem is proved.

Before finishing the paper, we mention that the condition $(l, k)=1$ is essential in the theorem, as the following example shows. Let $f(n)$ be the function defined by

$$f(n)=\begin{cases}\chi(n), & \text{if}\quad n\equiv 1\,(\mathrm{mod}\,2),\\ \chi(m), & \text{if}\quad n=2^{\alpha}m,\quad m\equiv 1\,(\mathrm{mod}\,2),\end{cases}$$

where $\chi(n)$ stands for the non-principal character mod 4. It is easy to see that the series

$$\sum_{\substack{n=1\\ n\equiv 2\,(\mathrm{mod}\,4)}}^{\infty}\frac{f(n)}{n},$$

being an alternating series of Leibniz type, converges and so for its partial sums

$$\sum_{\substack{n\leqq x\\ n\equiv 2\,(\mathrm{mod}\,4)}}\frac{f(n)}{n}=O(1)$$

holds. On the other hand it follows from the construction that

$$\sum_{\substack{n\leqq x\\ n\equiv 1\,(\mathrm{mod}\,4)}}f(n)=\sum_{\substack{n\leqq x\\ n\equiv 1\,(\mathrm{mod}\,4)}}\chi(n)=\sum_{\substack{n\leqq x\\ n\equiv 1\,(\mathrm{mod}\,4)}}1=\frac{x}{4}+O(1)\neq o(x).$$

References

[1] G. H. Hardy—E. M. Wright, *An introduction to the theory of numbers* (Oxford, 1954).
[2] E. M. Wright, Elementary proof of the prime number theorem, *Proc. Edinburgh Math. Soc.*, **63** (1951), 257—267.
[3] А. Г. Постников—Н. П. Романов, Упрощение элементарного доказательства А. Сельберга асимптотического закона распределения простых чисел, *Успехи Мат. Наук*, **10 (66)** (1955), 75—87.
[4] A. Selberg, An elementary proof of the prime number theorem for arithmetic progressions, *Canad. J. Math.*, **2** (1950), 66—78.

(Received August 15, 1966)