Minimal spectral sets of compact operators

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1. Introduction

In his paper [9], VON NEUMANN introduced the notion of a spectral set for an operator T on a Hilbert space. He proved that each spectral set is a superset of a *minimal* spectral set, but aside from the trivial case in which the spectrum of T is spectral, there are no other known minimal spectral sets. In the present paper we obtain a necessary and sufficient condition for the minimality of certain spectral sets of finite-dimensional (or compact) operators. A corollary is that the disk $|z| \leqq |T|$ is a minimal spectral set if T is compact and completely non-normal. An example shows that the result is not true without the adjective "compact". A few other results, based on an interpretation of VON NEUMANN's work as an extension of the Schwarz lemma, are also included.

2. Preliminaries

We begin with a summary of the relevant results of [9].

If T is a (bounded linear) operator on a complex Hilbert space, a closed set X is a *spectral set* of T if X contains the spectrum $\sigma(T)$ of T and if

$$\|u(T)\| \leqq \|u\|_X = \sup \{|u(z)| : z \in X\}$$

for each rational function $u(z)$ with poles off X. Any closed superset of a spectral set is again spectral, and, less trivially, any spectral set contains a *minimal* spectral set, i. e., a spectral set no proper closed subset of which is spectral. For example, if T is normal, then $\sigma(T)$ is a minimal spectral set of T. (A result of HALMOS implies the same conclusion if T is merely subnormal [4].) There is exactly one spectral set of T, or, equivalently, $\sigma(T)$ is spectral for T, if and only if the intersection of any two spectral sets of T is spectral. In general, the spectrum of T is not big enough to be spectral for T. Thus if X is spectral for T and X is "thin" in the sense that rational functions with poles off X are uniformly dense in the continuous functions on X, then T must be normal.

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[1]) The results of this paper constitute a portion of the author's thesis written under the supervision of Professor ARLEN BROWN at The University of Michigan.

The disk $|z| \leqq \|T\|$ is always a spectral set of T. This result is equivalent to the assertion that a half-plane is a spectral set of any T whose *numerical range* $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ is contained in that half-plane.

Finally, let us remark that as a consequence of the identical relation $\|(T-\alpha I)x\|^2 - \|(I-\bar{\alpha}T)x\|^2 = (1-|\alpha|^2)[\|T\alpha\|^2 - \|x\|^2]$, we have for $\varphi_\alpha(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}$ $(|\alpha| < 1)$:

$$\|\varphi_\alpha(T)\| \leqq 1 \quad \text{if} \quad \|T\| \leqq 1, \quad \text{and} \quad \|\varphi_\alpha(T)\| = 1 \quad \text{if} \quad \|T\| = 1.$$

3. Functions of a contraction

The classical Schwarz lemma is concerned with functions analytic in the open unit disk D. If we identify each $z \in D$ with the operator $z \cdot I$ we obtain a natural embedding of D into the set $\mathscr{D}$ of proper contractions on the Hilbert space H. It is reasonable to expect that the conclusion of the lemma is valid for all $T \in \mathscr{D}$. This is essentially what VON NEUMANN proved.

Theorem 1. (Schwarz lemma.) *Let H be a Hilbert space, $\mathscr{D}$ the set of proper contractions on H. If f is analytic in D, $f(0) = 0$ and $\|f\|_D \leqq 1$, then $\|f(T)\| \leqq \|T\|$ for each $T \in \mathscr{D}$. Moreover, equality can hold for some $T_0 \in \mathscr{D}$ only if $f(T) \equiv \gamma \cdot T$ for some constant γ of modulus* 1.

Proof. Note that if $T \in \mathscr{D}$, then f is analytic in a neighborhood of $\sigma(T)$ so that there is no difficulty in defining $f(T)$. Now if $T \in \mathscr{D}$, then by VON NEUMANN's theorem we have

$$\|f(T)\| \leqq \sup\{|f(z)| : |z| \leqq \|T\|\}$$

and since $f(0) = 0$, the usual version of the Schwarz lemma implies that the right member of this inequality is $\leqq \|T\|$. Moreover, we can have $\|f(T)\| = \|T\|$ for some $T \in \mathscr{D}$ only if there is a z_0 with $|z_0| = \|T\| < 1$ and $|f(z_0)| = \|T\|$. This occurs only if $f(z) \equiv \gamma \cdot z$ for some constant γ of modulus 1.

Corollary 1. *Let T be an operator, X a closed set containing $\sigma(T)$, and let μ be an interior point of X. If $\|u(T)\| \leqq \|u\|_X$ for each rational function which vanishes at μ, then X is spectral for T.*

Proof. Let v be a rational function with $\|v\|_X = 1$. We claim that $\|v(T)\| \leqq 1$. If $v(\mu) = 0$, this is true by hypothesis, otherwise $v(\mu) = \alpha$ has modulus less than 1 by the maximum principle, so that $\varphi_\alpha(z)$ is a conformal map of the disk D onto itself. Then $u(z) = \varphi_\alpha(v(z))$ is rational, vanishes at μ, and has bound 1 on X. Hence $u(T)$ is a contraction. But then, so is $v(T) = \varphi_\alpha^{-1}(u(T))$.

Corollary 2. *Consider the "two-dimensional shift" A_2 whose matrix relative to an orthonormal basis is*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\|\alpha + \beta A_2\| = \frac{1}{2}\{|\beta| + \sqrt{4|\alpha|^2 + |\beta|^2}\}$.

Proof. We have $\|\varphi_\alpha(A_2)\|=1$ for all $|\alpha|<1$. Since

$$\varphi_\alpha(A_2) = (A_2-\alpha)(1-\bar{\alpha}A_2)^{-1} = (A_2-\alpha)(1+\bar{\alpha}A_2) = -\alpha+(1-|\alpha|^2)A_2,$$

this gives

$$\|A_2-\alpha(1-|\alpha|^2)^{-1}\| = (1-|\alpha|^2)^{-1}.$$

Put $\lambda=\alpha(1-|\alpha|^2)^{-1}$ and compute

$$(1-|\alpha|^2)^{-1} = \frac{1}{2}\{1+\sqrt{4|\lambda|^2+1}\}$$

to arrive at

$$\|A_2-\lambda\| = \frac{1}{2}\{1+\sqrt{4|\lambda|^2+1}\}.$$

This yields the proposition for $\beta\neq 0$. If $\beta=0$, the result is trivial.

According to the Sz.-Nagy—Foiaş theory of contractions, a contraction T is *completely non-unitary* if T has no reducing subspace restricted to which T is unitary. A compact contraction is completely non-unitary if and only of its spectrum lies in D. (This follows from the fact that if $|\lambda|=\|A\|$, then $Ax=\lambda x$ is equivalent to $\langle Ax, x\rangle=\langle\lambda x, x\rangle$ and therefore to $A^*x=\bar{\lambda}x$.) The (unique, strong, minimal) unitary dilation $U=\int\lambda\, dE_\lambda$ of T has spectrum equal to the unit circle ∂D, and for each $x\neq 0$ in the domain of T, Lebesgue measure on the circle is equivalent to the measure $\langle E(\cdot)x, x\rangle$ [6].

Theorem 2. *Let T be a compact completely non-unitary contraction. If f is analytic in $|z|<1$ and bounded by 1 there, then $\|f(T)\|=1$ only if f is an inner function.*

Proof. First of all, since T is completely non-unitary, the operator $f(T)$ is well-defined for each bounded analytic function and for x, y in H,

$$\langle f(T)x, y\rangle=\langle f(U)x, y\rangle$$

where U is the unitary dilation of T [7]. Secondly, it is elementary that any compact operator attains its bound.

Now suppose that $f(T)$ has norm 1, where f is analytic and of bound 1 in $|z|<1$. Replacing $f(z)$, if necessary, by $\varphi_\alpha(f(z))$, where $\alpha=f(0)$, we may suppose that $f(0)=0$. Then $f(T)$ is compact and so we can choose a unit vector x in H so that $f(T)x$ has norm 1. Then

$$1 = \|f(T)x\|^2 \leqq \|f(U)x\|^2 = \int |f(\lambda)|^2 d\langle E_\lambda x, x\rangle.$$

Since $|f(\lambda)|\leqq 1$ on ∂D, this implies that $|f(\lambda)|$ has the value 1 on ∂D almost everywhere with respect to $\langle E(\cdot)x, x\rangle$. By the result referred to above, this in turn implies that f has modulus 1 on ∂D almost everywhere with respect to Lebesgue measure. This however is exactly the requirement the f be an inner function.

Remark. Using a result of HAVINSON [3, Theorem 2. 5] a stronger version of Theorem 2 can be obtained. Under the same hypothesis on f and T one can show that f is a Blaschke product with only finitely many zeros.

4. Completely non-normal operators

We call an operator T *completely non-normal* if T has no reducing subspace restricted to which T is normal. Any operator T is the direct sum of a normal operator N and a completely non-normal operator T_0 (*cf.* e. g. [10]). It follows that a closed set X is spectral for T if and only if X is spectral for T_0 and X contains $\sigma(N)$, so the theory of spectral sets reduces to the study of completely non-normal operators.

If T is completely non-normal and $\psi(z)=(az+b)(cz+d)^{-1}$ is analytic on $\sigma(T)$, then $\psi(T)$ is also completely non-normal. Hence the class of completely non-normals is closed with respect to translation, inversion, adjunction, and scalar multiplication. It is not however closed with respect to products or sums.

If $\dim H \geqq 3$, the notions of completely non-normal and completely non-unitary are distinct. If H is two-dimensional there are essentially only two completely non-normal operators as the following theorem shows.

Theorem 3. *Let T be a completely non-normal operator of norm* 1 *on a two-dimensional space. Either $T=\alpha+\beta A_2$ with $|\beta|=1-|\alpha|^2$, or there exist unit vectors x_1, x_2 and scalars λ_1, λ_2 such that $Tx_i=\lambda_i x_i$. In the second case the following relations are valid:*

(i) $0<|\langle x_1, x_2\rangle|<1$,

(ii) $\sqrt{1-|\langle x_1, x_2\rangle|^2} = |(\lambda_2-\lambda_1)(1-\bar{\lambda}_1\lambda_2)^{-1}|$.

Observe that if T is the two-dimensional completely non-normal operator with two eigenvalues and norm 1, then for any function f analytic in D, the above theorem implies that $f(T)$ has norm 1 if and only if

$$|(f(\lambda_2)-f(\lambda_1))(1-\overline{f(\lambda_1)}f(\lambda_2))^{-1}| = |(\lambda_2-\lambda_1)(1-\bar{\lambda}_1\lambda_2)^{-1}|.$$

Consequently, if f also has bound 1 on D, then the Schwarz lemma implies that f is a conformal map of D onto itself. It is easy to see that the same conclusion is also valid for the other two-dimensional completely non-normal operator. We record these facts as a

Corollary. *Let T be a completely non-normal operator of norm* 1 *on a two-dimensional space. Then the only non-constant functions analytic in D which satisfy*

$$\|f(T)\|=\|f\|_D=1$$

are the conformal maps of D onto itself.

We conclude this section with a decomposition theorem which, although it does not appear in the literature, is probably known to specialists. Consider a compact set X containing the spectrum of an operator T, and suppose that X is the union of two non-empty disjoint sets X_1 and X_2. Put

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda-z}$$

where γ is a rectifiable path surrounding X_1 and containing X_2 in its exterior. Then f is analytic on X, identically 1 on X_1 and identically zero on X_2. Put $E=f(T)$.

Theorem 4. *If X is spectral for T, then E is a self-adjoint projection, the range of E reduces T, and X_1 is spectral for the restriction of T to $E(H)$.*

Proof. The operator E is idempotent and commutes with T [2] so that it suffices to show that E is self-adjoint and that X_1 is spectral for $T|E(H)$.

The first assertion is almost trivial. Thus by approximating the integral defining f, one sees that f is the uniform limit on X of functions which are rational and bounded on X, hence

$$\|E\| = \|f(T)\| \leqq \|f\|_X = 1.$$

It remains only to observe that an idempotent of norm 1 is necessarily self-adjoint; this is straightforward and we omit the details.

It now follows that $H_1 = E(H)$ reduces T. To prove that X_1 is spectral for $T_1 = T|H_1$, let u be a rational function with poles off X_1. Set

$$v(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(\lambda)}{\lambda - z} d\lambda$$

where γ is the path used to define E, and note that $v(z)$ is a uniform limit of functions which are rational and bounded on X. Also

$$v(z) = \begin{cases} u(z) & z \in X_1, \\ 0 & z \in X_2, \end{cases}$$

and so

$$\|v(T)\| \leqq \|v\|_X = \|u\|_{X_1}.$$

Now use the fact that $u(T_1) = v(T)|H_1$ (see [2, p. 574] for example) to conclude that $\|u(T_1)\| \leqq \|u\|_{X_1}$. Thus X_1 is spectral for T_1.

Corollary 1. *If the spectrum of T consists of a single point, in particular, if T is quasi-nilpotent, then each minimal spectral set of T is connected. The same conclusion is valid for any completely non-normal operator if* dim $H \leqq 3$, *but is otherwise false.*

Proof. The first assertion is clear, and the second follows from the fact that if E is a self-adjoint projection with one-dimensional range, and if E commutes with T, then T is normal on the range of E.

To complete the proof observe that the set

$$X = \{z: |z| \leqq 1\} \cup \{z: |z-3| \leqq 1\}$$

is not connected, and is spectral for the completely non-normal operator $T = A_2 \oplus \oplus (A_2 + 3)$. We will prove later that X is in fact a minimal spectral set for T.

Corollary 2. *If the operator T is irreducible, then each minimal spectral set of T is connected.*

There is another consequence of Theorem 4 which J. Stampfli pointed out to me. To state this, recall that an operator T on H is *subnormal* if T is the restriction to H of a normal operator N acting on a space $K \supset H$.

Corollary 3. *If T is subnormal and if $\sigma(T)=\sigma_1 \cup \sigma_2$ with σ_1 and σ_2 non-empty and disjoint, then T can be decomposed into a direct sum $T=T_1 \oplus T_2$ of subnormal operators with $\sigma(T_i)=\sigma_i$ $(i=1, 2)$.*

Proof. The spectrum of a subnormal operator is spectral [4].

In the next section we will be interested in determining which spectral sets are minimal. The general problem of course reduces to the case of a completely non-normal operator. By further restricting attention to irreducible operators, the preceding corollary allows us to consider only connected spectral sets. It is this latter problem we will study, not in complete generality but with the additional assumption that the sets in question have nice boundaries.

5. Minimal spectral sets of completely non-normal operators

In the remainder of this paper G will denote a bounded region (open, connected set) in the plane; G has finite connectivity n, and the boundary ∂G of G is the union of n disjoint, closed, rectifiable Jordan curves. We assume that these curves are oriented in the usual positive sense with respect to G.

$B(G)$ will denote the algebra of functions analytic and bounded in G; $B_1(G)$ consists of those $f \in B(G)$ whose norm $\|f\|=\|f\|_G=\sup\{|f(z)|: z \in G\}$ does not exceed 1.

Our main result depends on a theorem of HAVINSON concerning extremal problems in the region G. To state this theorem we introduce the following definition $\left(\text{here } \partial G = \bigcup_{i=1}^{n} \gamma_i, \text{ and } \gamma_1 \text{ is the outer boundary of } G\right)$.

Definition [3]. Let f be analytic in G. Then $f \in E_p(G)$ $(p>0)$ if there is a sequence of closed rectifiable Jordan curves $\Gamma^k = \bigcup_{i=1}^{n} \gamma_i^k$ such that

(1) γ_1^k lies inside γ_1 and γ_i^k $(i=2, \ldots, n)$ contains γ_i inside it for every k,
(2) $\gamma_i^k \to \gamma_i$ as $k \to \infty$ $(i=1, 2, \ldots, n)$,
(3) the lengths of the Γ^k are uniformly bounded,
(4) $\sup\limits_{k} \int_{\Gamma^k} |f(z)|^p |dz| < \infty$.

The space $E_p(G)$ is obviously a generalization of the classical space H^p to multiply connected regions, and most of the classical results have analogues in $E_p(G)$. For example, any E_p function has boundary values (for approach in an angle) almost everywhere with respect to arc length, and the function itself can be written as the Cauchy integral of its boundary values. The F. and M. Riesz Theorem is also valid in E_p: If $f \in E_p$ has 0 boundary values on a set of positive arc length then f must vanish identically.

Theorem 5. (HAVINSON) *Let $\omega(\lambda)$ be summable on $\Gamma=\partial G$. Then*

(1) $$\sup_{f \in B_1(G)} \left| \int_{\Gamma} f(\lambda)\,\omega(\lambda)\,d\lambda \right| = \inf_{\Phi \in E_1(G)} \int_{\Gamma} |\omega(\lambda)-\Phi(\lambda)|\,ds.$$

(2) *The infimum on the right is always attained by an extremal function $\Phi \in E_1(G)$.*

(3) *The supremum on the left is attained by a function* $f \in B_1(G)$ *and, moreover,* f *is unique to within a factor* $e^{i\alpha}$ *provided* $\omega(\lambda)$ *is not the boundary value of any* E_1 *function.*

(4) *A necessary and sufficient condition for* $f \in B_1$ *and* $\Phi \in E_1$ *to be extremal functions is that almost everywhere on* Γ,

$$f(\lambda)[\omega(\lambda) - \Phi(\lambda)]\,d\lambda = e^{i\alpha}|\omega(\lambda) - \Phi(\lambda)|\,ds$$

where α *is a real constant.*

To conclude these general considerations, consider the situation in which our region G contains the spectrum of an operator T. Then for any $g \in B(G)$ we can form the operator $g(T)$ by the Riesz—Dunford functional calculus. If g is actually analytic on $\bar{G}$, then $g(T)$ is given by the integral

$$\langle g(T)x, y\rangle = -\frac{1}{2\pi i}\int_\Gamma g(\lambda)\langle R_\lambda(T)x, y\rangle\,d\lambda \qquad (x, y \in H)$$

The same formula is valid for any $g \in B(G)$. This is a consequence of the following facts:

(1) If $g \in B(G)$, then $g = \lim g_n$ where the g_n are uniformly bounded and analytic on $\bar{G}$, and the limit is subuniform in G.

(2) If $\{g_n\}$ is a sequence of uniformly bounded analytic functions which converges to g subuniformly in G, then for any function $\omega(\lambda)$ summable on Γ

$$\int_\Gamma g_n(\lambda)\omega(\lambda)\,d\lambda \to \int_\Gamma g(\lambda)\omega(\lambda)\,d\lambda \quad \text{(see [3])}.$$

We begin now the task of applying the preceding function theory to the study of the minimal spectral sets of a fixed completely non-normal operator T on a finite-dimensional Hilbert space (the infinite case will be discussed later). The connection is made possible by the fact that $\bar{G}$ can be spectral for T only if $\sigma(T)$ is wholly contained in G:

Theorem 6. (Sz.-Nagy—Foiaş [8]) *Let* S *be the closure of a simply connected region bounded by a Jordan curve, and assume that* S *is spectral for an operator* A. *If* $\lambda \in \partial S$, *then* $Ax = \lambda x$ *if and only if* $A^*x = \bar{\lambda}x$.

Corollary. *If* $\bar{G}$ *is spectral for* T, *then* $\sigma(T) \subset G$.

Proof. Write $G = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n$, and let $G_1, G_2, \ldots, G_n$ be the components of the complement of $\bar{G}$. We may assume that $\infty \in G_1$. Then the complement G_1' is the closure of a simply connected region bounded by the Jordan curve γ_1, and moreover, G_1' is spectral for T because $G_1' \supset \bar{G}$. Hence by Theorem 6, $\sigma(T)$ cannot meet γ_1.

If $2 \leq k \leq n$, choose $z_k \in G_k$ and let $\omega_k(z) = (z - z_k)^{-1}$. Then $\varphi_k(T)$ is completely non-normal and $\varphi_k(G_k)$ is spectral for $\varphi_k(T)$. Applying the theorem we conclude that $\sigma(\varphi_k(T)) = \varphi_k(\sigma(T))$ does not meet $\varphi_k(\gamma_k)$, that is $\sigma(T) \cap \gamma_k = \varphi$.

Remark. The proof of Theorem 6 given in [8] uses unitary dilations, but there is a more elementary proof. Thus if S is the unit disk the result is a consequence of the fact that if $\gamma \in \partial S$ then both $Ax = \lambda x$ and $A^* x = \bar{\lambda} x$ are equivalent to $\langle Ax, x\rangle = \lambda\langle x, x\rangle$. In the general case the set S is the image of $\bar{D}$ under a function f which is $1-1$ and continuous on $\bar{D}$, analytic in D. Then f^{-1} is the limit of a sequence of polynomials which converge uniformly on S and because S is spectral for A, $f^{-1}(A)$ is a contraction. The assertion about A and λ then reduces to the same assertion about $f^{-1}(A)$ and $f^{-1}(\lambda)$.

Remark 2. The preceding corollary is not valid under the assumption that T is completely non-unitary. For example, $T = A_2 \oplus \lambda_0 I$ is completely non unitary if $\lambda_0 = \|A_2 + 1\| - 1$. Also λ_0 belongs to $\sigma(T)$ and λ_0 lies on the boundary of the spectral set

$$S = \{z\colon |z+1| \leqq \|A_2 + 1\|\}.$$

(Incidentally, the same example is the basis for our earlier remark that the notions of completely non-normal and completely non-unitary are distinct as soon as $\dim H \geqq 3$.)

Theorem 7. *Let T be completely non-normal and let G be rectifiably bounded as above. Then $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,

(2) $\max\{\|f(T)\| : f \in B_1(G)\} \leqq 1$.

If $\mu \in G$ is fixed, then (2) *is equivalent to*

(2′) $\max\{\|f(T)\| : f \in B_1(G), f(\mu) = 0\} \leqq 1$.

Proof. We have just seen that (1) is necessary. The necessity of (2) follows from the facts that (a) the rational functions with poles off $\bar{G}$ are subuniformly dense in $B(G)$ and (b), $B_1(G)$ is a compact subset of $B(G)$ for this topology so that the continuous functional $f \to \|f(T)\|$ attains its supremum over $B_1(G)$. The sufficiency of (1) and (2) are obvious, and the equivalence of (2) and (2′) is a standard application of Corollary 2 of Theorem 1.

The next result gives a necessary condition for certain spectral sets to be minimal. (Here the operator A is quite arbitrary.)

Theorem 8. *Let X be a minimal spectral set for the operator A and assume that $\sigma(A) \subset \operatorname{int} X$. If $\mu \in \operatorname{int} X$, then there is a function f analytic in* int X *with $f(\mu) = 0$ and*

$$\|f(A)\| \geqq \sup\{|f(z)| : z \in \operatorname{int} X\}.$$

Proof. We may assume that int X is connected. Choose a point $a \in \operatorname{int} X$ with $a \notin \sigma(A) \cup \{\mu\}$ and put

$$X_n = \{z\colon |z-a| \geqq 2^{-n} d\} \cap \operatorname{int} X \qquad (n = 1, 2, \ldots)$$

where d is the distance from a to $\sigma(A) \cup \{\mu\}$. Then $\sigma(A) \subset \operatorname{int} X_n$ and $X_n \subsetneqq X$. Since X is minimal for A, it follows that $\bar{X}_n$ is not spectral for A, and since $\mu \in \operatorname{int} X_n$, this in turn implies that there is a rational function $u_n(z)$ with poles off $\bar{X}_n$ such that

$$u_n(\mu) = 0, \ \|u_n(A)\| > \|u_n\|_{X_n} = 1.$$

Now the functions $u_n(z)$ form a normal family in int X minus the point a, and hence a subsequence u_{n_k} converge to a limit function f uniformly on compact subsets.

It is easy to see that a is a removable singularity of f and hence f extends to be analytic in int X. Then f is bounded by 1 vanishes at $z=\mu$ and

$$\|f(A)\| = \lim_k \|u_{n_k}(A)\| \geqq 1$$

because $u_{n_k} \to f$ uniformly on $\sigma(A)$.

We are finally able to prove the principal theorem.

Theorem 9. *Let G be rectifiably bounded and let T be completely non-normal as above. Fix $\mu \in G$. Then $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,

(2) $\max\{\|f(T)\| : f \in B_1(G), f(\mu)=0\} \leqq 1$.

Moreover, $\bar{G}$ is minimal for T if and only if this maximum is 1.

Proof. In view of Theorems 7 and 8 remains only to prove the sufficiency of the minimality condition. For this, let $f \in B_1(G)$ be an extremal function with $f(\mu)=0$ and $\|f(T)\| = \|f\|_G = 1$. Since the underlying space is finite-dimensional we can choose unit vectors x and y so that $\langle f(T)x, y\rangle = 1$. Now any proper closed subset of $\bar{G}$ which contains $\sigma(T)$ fails to contain some point of G and hence is contained in a set of the form

$$S = \{z \in \bar{G}: |z-a| \geqq \varepsilon\} \qquad (a \in G_1, \varepsilon > 0).$$

Hence to prove that $\bar{G}$ is minimal, it suffices to show that S is not spectral for T. We make use of the fact that $\partial S = \partial G \cup \gamma_{n+1}$ where γ_{n+1} is a circle contained in G.

If S is spectral for T, then it follows from Theorem 7 that f is an extremal function for the problem

$$\sup\left\{\left|\int_{\partial S} g(\lambda)\,\omega(\lambda)\,d\lambda\right|, \quad g \in B_1(\operatorname{int} S), \quad g(\mu) = 0\right\}$$

where $\omega(z) = \langle R_z(T)x, y\rangle$. By HAVINSON's theorem there is a function $\varphi \in E_1(\operatorname{int} S)$ such that almost everywhere on ∂S,

$$f(\lambda)(\omega(\lambda) - \Phi(\lambda))\,d\lambda = e^{i\alpha}|\omega(\lambda) - \Phi(\lambda)|\,ds$$

where α is a real constant. This in turn implies that $|f(\lambda)| = 1$ on the subset Z of ∂S consisting of those λ for which $\omega(\lambda) \neq \Phi(\lambda)$. Assuming for the moment that we can show that Z intersects γ_{n+1} it will then follow from the maximum principle that f is constant. However since f vanishes at μ and $\|f(T)\| = 1$, this is a contradiction. In short, we need the following

Lemma. *The function $\omega(z) = \langle R_z(T)x, y\rangle$ does not coincide on γ_{n+1} with a function of class $E_1(\operatorname{int} S)$.*

Proof. The spectrum of T consists of finitely many interior points of S and so we can choose a finite number of open disks D_i such that

$$\bar{D}_i \cap \bar{D}_j = \emptyset \quad \text{if} \quad i \neq j, \qquad \bar{D}_i \subset \operatorname{int} S, \qquad \sigma(T) \subset \bigcup D_i.$$

Let S_1 be the (rectifiably bounded) set obtained from S by deleting these disks and let δ be the boundary of $\bigcup D_i$. Then $\partial S_1 = \partial S \cup \gamma_{n+1} \cup \delta$.

Suppose now that $\omega=\Phi$ on γ_{n+1}, where Φ is a function of class $E_1(\text{int } S)$. Since ω is bounded on S_1, $\omega \in E_1(\text{int } S_1)$. Also $\Phi \in E_1(\text{int } S_1)$ and so by the Riesz theorem we can conclude that $\omega=\Phi$ on ∂S_1. In particular, $\omega=\Phi$ on δ. This however is impossible, for on one hand

$$\int_\delta f(\lambda)\omega(\lambda)\,d\lambda = 0$$

(because the integrand is analytic in $\bigcup D_i$), while on the other hand

$$\int_\delta f(\lambda)\omega(\lambda)\,d\lambda = 2\pi i \langle f(T)x, y\rangle \neq 0$$

because δ is a path in G surrounding $\sigma(T)$. The assumption $\omega=\Phi$ on γ_{n+1} therefore leads to a contradiction and the lemma is proved.

Corollary 1. *The unit disk is a minimal spectral set for any completely non-normal operator of norm* 1.

Proof. If $\mu \in \sigma(T)$, then $|\mu|<1$ and so the function $\varphi_\mu(z)=(z-\mu)(1-\bar{\mu}z)^{-1}$ belongs to $B_1(D)$, vanishes at $z=\mu$ and $\|\varphi_\mu(T)\|=1$. Since $\bar{D}$ is spectral for T, it follows from the theorem just proved that $\bar{D}$ is in fact minimal.

Corollary 2. (Converse of the Schwarz Lemma.) *Let X be a closed subset of the closed unit disk $\bar{D}$ which contains* 0. *If $|u'(0)| \leqq \|u\|_X$ for each rational function $u(z)$ which vanishes at* 0, *then $X=\bar{D}$. Similarly, if for some $\lambda \in X$ with $0<|\lambda|<1$ the conditions $u(z)$ rational, $u(0)=0$, $\|u\|_X=1$ imply $|u(\lambda)| \leqq |\lambda|$, then $X=\bar{D}$.*

Proof. The first assertion follows from the fact that $\bar{D}$ is a minimal spectral set for A_2. The second assertion follows similarly by considering the two-dimensional completely non-normal of norm 1 with eigenvalues 0 and λ.

Using the fact that linear fractional transformations preserve both complete non-normality and minimality of a spectral set we get the following improvement of VON NEUMANN's theorem:

Corollary 3. *Let T be completely non-normal. Then*

$$S_1=\{z\colon |z-\lambda| \leqq \|T-\lambda\|\} \quad \textit{and} \quad S_2=\{z\colon |z-\lambda| \geqq \|R_\lambda(T)\|^{-1}\}$$

are minimal spectral sets of T. The set

$$S_3=\{z\colon \operatorname{Re} z \geqq 0\}$$

is a minimal spectral set of T if either $\|(T-1)(T+1)^{-1}\|=1$ or if the numerical range of T lies in the right half-plane and meets the imaginary axis.

Remark. In the hypothesis of Theorem 9 it is actually superfluous to require that T be completely non-normal. Indeed, the assumption $\max\{\|f(T)\|: f \in B_1(G), f(\mu)=0\}=1$ implies that T has a nontrivial completely non-normal part. For if $T=N$ is normal with $\sigma(N) \subset \text{int } S$, and if $f \in B_1(G)$ is chosen as an extremal function for the problem

$$\max\{\|g(N)\|\colon g \in B_1(G),\ g(\mu)=0\}$$

then we have

$$\|f(N)\| = \sup\{|f(z)|: z \in \sigma(N)\} = \|f\|_{\sigma(N)}.$$

Since f is continuous on $\sigma(N)$ it attains its maximum there and the maximum principle then shows that $\|f\|_{\sigma(N)} < 1$.

6. Remarks on the infinite case

If T is a completely non-normal operator on an infinite-dimensional space, then our previous argument shows that both the point spectrum and residual spectrum of T are subsets of the *interior* of any Jordan spectral set of T. In general, this is best possible as the unilateral shift shows (point spectrum void, residual spectrum $= \{|z| < 1\}$, continuous spectrum $= \{|z| = 1\}$.) It is perhaps surprising that even compact operators can exhibit this behavior.

Example 1. The spectrum of a compact completely non-normal operator need not be contained in the interior of each Jordan spectral seet.

Consider the Volterra operator A defined on $L^2(0, 1)$ by

$$(Af)(t) = \int_0^t f(s)\, ds.$$

It is well known that A is compact with $\sigma(A) = \{0\}$ and Re $A \geqq 0$. It follows from the equality of norm and spectral radius for normal operators that A is completely non-normal. Now let $T = (1-A)(1+A)^{-1} - 1$ and observe that T is compact, completely non-normal, and $T+1$ is a contraction because

$$\|(T+1)x\|^2 - \|x\|^2 = \|(1-A)(1+A)^{-1}x\|^2 - \|x\|^2 =$$

$$= \|(1-A)y\|^2 - \|(1+A)y\|^2 = -4\,\mathrm{Re}\,\langle Ay, y\rangle \leqq 0 \qquad (y = (A+1)^{-1}x).$$

It follows that the unit disk $\bar{D}$ is spectral for $T+1$, and hence $S = \bar{D} - 1 = \{z: |z+1| \leqq 1\}$ is a Jordan spectral set for T. Finally, $\sigma(T) = \{0\}$ meets the boundary of S.

Example 1 shows that the techniques of § 5 are not suitable for establishing the minimality of rectifiably bounded spectral sets of arbitrary completely non-normal operators. It is natural to expect however that the results are extendable by other means. Even this is not possible:

Example 2. The unit disk $\bar{D}$ is not a minimal spectral set for every completely non-normal operator of norm 1.

Recall that if T is subnormal, then $\sigma(T)$ is a spectral set of T. It follows then that it is sufficient to exhibit a completely non-normal subnormal operator of norm 1 whose spectrum is a proper subset of $\bar{D}$. Our construction is motivated by the theory of analytic Toeplitz operators developed in [1].

First of all, it is easy to see that if T is subnormal on H with minimal normal extension N on $K \supset H$, then T is completely non-normal if and only if no non-trivial subspace of H reduces N.

Now to construct the example, let V be the unilateral shift on H^2 and U the bilateral shift on L^2. Let φ be a conformal map of $\bar{D}$ onto the half-disk $S=\{\text{Re}\, z \geqq 0\} \cap \bar{D}$. We claim that $T=\varphi(V)$ is subnormal, completely non-normal, and S is spectral for T.

The subnormality of T is clear: $T=\varphi(U)|H^2$ and $\varphi(U)$ is normal. To prove that T is completely non-normal it suffices, by the above remark, to show that any reducing subspace H_0 for $\varphi(U)$ which is contained in H^2 is trivial. But if H_0 reduces $\varphi(U)$ then $\varphi(U)$ commutes with the projection P of L^2 onto H_0 and this implies that U commutes with P, that is, H_0 is a reducing subspace of U contained in H^2. However, it is well-known that the only reducing subspace of U contained in H^2 is the subspace $\{0\}$. Consequently, T is completely non-normal.

Finally, S is spectral for T because $\sigma(T)$ is spectral and $S \supset \sigma(T)$. (if $\varphi(z)-\lambda$ is bounded below on $\bar{D}$, then $\varphi(V)-\lambda$ has an inverse.)

The preceding example indicates that it is not just complete non-normality of a contraction which forces the unit disk to be a minimal spectral set and it is therefore worth investigating our earlier arguments a little more carefully. Some of these are independent of the dimensionality of the underlying space. For example, if G is the rectifiably bounded region previously studied, and T is any operator with $\sigma(T) \subset G$, the condition for spectrality of $\bar{G}$ for T is still the same:

$$\max\,\{\|f(T)\| : f \in B_1(G), f(\mu)=0\} \leqq 1$$

and if $\bar{G}$ is minimal for T, this maximum is 1. There are two difficulties encountered in proving the sufficiency of the minimality condition. In the first place we need the fact that certain functions of T attain their bound. The second difficulty concerns the proof of the Lemma of § 5 where we explicitly assumed that $\sigma(T)$ was a finite set. The latter requirement is not really essential and it is easy to extract the following extension of Theorem 9:

Theorem 10. *Let G be rectifiably bounded as before and let $0 \in G$. Let T be a compact completely non-normal operator. $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,
(2) $\max\,\{\|f(T)\| : f \in B_1(G), f(0)=0\} \leqq 1$.

Moreover, $\bar{G}$ is minimal for T precisely when the maximum is 1.

Proof. If T is compact, then so is any $f(T)$ with $f(0)=0$ and so $f(T)$ attains its bound. Secondly, if S is a rectifiably bounded subset of $\bar{G}$ whose boundary meets G, then because 0 is the only limit point of $\sigma(T)$ we can still punch finitely many holes in int S to get the set S_1 needed in the proof of the lemma of § 5. The remainder of the argument is exactly as before.

Corollary 1. *If T is compact and completely non-normal, then $|z| \leqq \|T\|$ is a minimal spectral set of T.*

There is an obvious extension of the corollary: If T has norm 1, is completely non-normal and for some α of modulus less than 1 the operator $\varphi_\alpha(T)$ is compact $\left(\varphi_\alpha(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}\right)$, then $\bar{D}$ is minimal for T. To prove this note that $\varphi_\alpha(T)$ has norm 1, is completely non-normal and so by the above corollary, $\bar{D}$ is minimal for T. This implies that $\varphi_\alpha^{-1}(\bar{D})=\bar{D}$ is minimal for $\varphi_\alpha^{-1}(\varphi_\alpha(T))=T$.

Since $T-\alpha=\varphi_\alpha(T)(1-\bar{\alpha}T)$, the operator $\varphi_\alpha(T)$ is compact precisely when $T-\alpha$ is compact, and so the extension of the corollary reads as follows:

Corollary 2. *Let T be completely non-normal of norm 1 and suppose that $T-\alpha$ is compact for some α of modulus less than 1. Then $\bar{D}$ is minimal for T.*

The signifiance of the condition $|\alpha|\leqq 1$ is clear: $T-\alpha$ can be compact only if $0\in\sigma(T-\alpha)$, i. e., only if $\alpha\in\sigma(T)$. What is not clear however is that the condition fails in case $|\alpha|=1$. Equivalently, if T has norm 1, is completely non-normal, and $T-1$ is compact, must $\bar{D}$ be minimal for T?

7. An application concerning numerical ranges

Recently several authors have been interested in the relation between the spectral sets of an operator T and its numerical range $W(T)$. VON NEUMANN's theorem asserts that a closed half-plane H is spectral for T if and only if $H\supset W(T)$. The latter inclusion is equivalent to

$$\|(T-\lambda)^{-1}\|\leqq\sup\{|z-\lambda|^{-1}: z\in H\}\qquad(\lambda\notin H)$$

and so to determine whether or not H is spectral we need only look at (a subset of) the rational functions of order 1 with poles off H. This fact leads naturally to the question of whether one can similarly prescribe a sub-class of rational functions which determine the spectrality of $\overline{W(T)}$. Such a result is the following one:

If $\|p(T)\|\leqq\|p\|_{W(T)}$ for all polynomials, then $\overline{W(T)}$ is a spectral set for T.

(Proof. The compact set $\overline{W(T)}$ has a connected complement and so by a theorem of LAVRENTIEFF (see [5]) the polynomials are uniformly dense in the algebra of functions which are continuous on $\overline{W(T)}$ and analytic in int $\overline{W(T)}$.)

It is an elementary fact that if $|W(A)|$ and $|\sigma(A)|$ denote the numerical radius and the spectral radius of an operator A, respectively, one has $\|A\|=|W(A)|$ if and only if $\|A\|=|\sigma(A)|$. It follows that either of the equivalent conditions

$$\|p(T)\|=|\sigma(p(T))|,\quad \|p(T)\|=|W(p(T))|$$

(for all polynomials) is a sufficient condition for the spectrality of $\overline{W(T)}$. It seems reasonable to ask whether the condition remains sufficient when the class of *all* polynomials is replaced by the linear ones. That is, does the condition $|W(T-\lambda)|=$ $=\|T-\lambda\|$ (all complex λ) imply spectrality of $\overline{W(T)}$? The following example answers the question negatively.

Example 3. Let $T=A_2\oplus U$ where U is unitary with spectrum equal to eth cube roots of unity. Then $|W(T-\lambda)|=\|T-\lambda\|$ for all complex λ but $\overline{W(T)}$ is not spectral for T.

First of all, $W(T)$ is the convex hull of the numerical ranges of $W(A_2)$ and $W(U)$, and hence $W(T)$ is the equilateral triangle which constitutes $W(U)$. Therefore

$$\|U-\lambda\|=|W(U-\lambda)|=|W(T-\lambda)|,$$

thus

$$\|T-\lambda\|=\max\{\|A_2-\lambda\|,\|U-\lambda\|\}=\max\{\|A_2-\lambda\|;|W(T-\lambda)|\}.$$

It remains to see that

(1) $\|A_2-\lambda\| \leqq |W(T-\lambda)|$.

(2) The triangle $W(U)$ is not spectral for A_2.

The first of these is a simple computation (see Corollary 2 of Theorem 1), and the second is a consequence of the fact that the unit disk is a minimal spectral set for A_2.

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