On the order of magnitude of the partial sums of rearranged Fourier series of square integrable functions

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Introduction

KOLMOGOROFF [1] was the first to remark that there exists a square integrable function the Fourier series of which diverges almost everywhere in a certain rearrangement of its terms. However, he has never published the proof of this fact. Afterwards ZAHORSKI [2] sketched a proof of this assertion. Recently OLEVSKIĬ [3] and UL'JANOV [4] obtained some more general theorems. Then, using less elementary tools, TAĬKOV [5] obtained a somewhat sharper result, and a direct elementary construction leading to KOLMOGOROFF's assertion was given by TANDORI [6]. In this paper we are going to sharpen this result by refining a method, due to TANDORI [7], concerning the rearrangement of Walsh series.

UL'JANOV has raised the following question [4]: what is the exact Weyl multiplicator of unconditional convergence in case of Fourier series? We shall show that it is at least $O(\log\log n)$. [1])

Theorem 1. *If $\{\varrho(n)\}$ is any sequence of positive real numbers for which*

$$\varrho(n) = o\left(\sqrt{\log\log n}\right) \tag{1}$$

is satisfied, then there exists a square integrable function whose Fourier series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is such that

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\varrho^2(n) < \infty \tag{2}$$

and which can be rearranged into an everywhere divergent series

$$\sum_{j=1}^{\infty} \left(a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x\right).$$

For the partial sums of the rearranged Fourier series we have following estimate:

[1]) In this paper log means logarithm with base 4 (but this is not essential to our results).

Theorem 2. *If $\{\varrho(n)\}$ is any sequence of positive real numbers for which* (1) *holds, then there exists a square integrable function whose Fourier series*

$$\sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

can be rearranged in a such a way that the partial sums $\sigma_N(x)$ of the rearranged series

$$\sum_{j=1}^{\infty} (A_{n(j)} \cos n(j)x + B_{n(j)} \sin n(j)x)$$

satisfy

$$\limsup_{N\to\infty} \frac{|\sigma_N(x)|}{\varrho(N)} > 0 \tag{3}$$

everywhere.

I am grateful to Professor KÁROLY TANDORI for calling my attention to this problem.

§ 1. Lemmas

Consider the Fejér kernel

$$K_n(x) = \frac{1}{2(n+1)} \left(\frac{\sin (n+1) \frac{x}{2}}{\sin \frac{x}{2}} \right)^2 = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx.$$

By a simple calculation we obtain the following inequalities

$$K_n(x) \geqq \frac{2(n+1)}{\pi^2} \quad \text{if} \quad |x| \leqq \frac{\pi}{n+1}, \tag{4}$$

$$K_n(x) \leqq \frac{n+1}{2}, \quad K_n(x) \leqq \frac{\pi^2}{2(n+1)x^2} \quad \text{if} \quad |x| \leqq \frac{\pi}{2}, \tag{5}$$

and

$$\int_{-\pi}^{\pi} K_n^2(x)\, dx < \pi n. \tag{6}$$

In the following a set E will be said to be *simple* if it is the union of finitely many, non-overlapping, closed intervals $[\alpha_k, \beta_k]$ $(\alpha_k < \beta_k)$. For any $\varepsilon > 0$ $(\varepsilon < \min_k (\beta_k - \alpha_k)/2)$, we set

$$E^{(\varepsilon)} = \bigcup_k [\alpha_k + \varepsilon, \beta_k - \varepsilon].$$

For a function $a_\nu \cos \nu x + b_\nu \sin \nu x$ $(\not\equiv 0)$ we call ν its *frequency*. Two trigonometric polynomials will be called *disjoint* if they have no terms of the same frequency.

$C_1, C_2, \ldots$ will denote positive absolute constants.

Lemma 1. *Let $\delta(\leqq\pi/8)$, $\varepsilon(<\delta)$ and $\eta(\leqq 1)$ be positive real numbers, and let n be a natural number such that $n>C_1/\varepsilon\eta$. Then there exists a non-negative trigonometric polynomial $P(x)$ with frequencies 4ν $(\nu=0, 1, \dots, n)$ and having the following properties:*

(7) $$P(x)\geqq 1 \quad \text{if} \quad |x|\leqq\delta-\varepsilon,$$

(8) $$P(x)\leqq\eta \quad \text{if} \quad \delta\geqq|x|\leqq\frac{\pi}{8},$$

and

(9) $$\int_{-\pi}^{\pi} P^2(x)\,dx \leqq C_2\delta.$$

Proof. We write

$$a=\frac{\pi}{4(n+1)}, \quad b_k=\frac{k\pi}{2(n+1)} \quad (k=0, \pm 1, \pm 2, \dots).$$

Let the integers ϱ and σ be determined by the inequalities

$$b_\varrho-a\leqq-\delta+\varepsilon<b_\varrho \quad \text{and} \quad b_\sigma<\delta-\varepsilon\leqq b_\sigma+a.$$

This choice of ϱ and σ is possible because $n+1>\pi/2\varepsilon$.

Define the trigonometric polynomial $P(x)$ by

$$P(x)=2\pi a\sum_{r=\varrho}^{\sigma} K_n(4(x-b_r)).$$

We are going to show that $P(x)$ has the properties (7)—(9). On account of the choice of ϱ and σ and (4) we can easily see that (7) is satisfied.

To prove the inequality (8), suppose $\delta\leqq|x|\leqq\pi/8$. Using (5), it follows

$$P(x)\leqq 2\pi a\sum_{r=\varrho}^{\sigma}\frac{\pi^2}{32(n+1)(x-b_r)^2}<\frac{\pi^3 a}{16(n+1)}\sum_{k=0}^{\infty}\frac{1}{(\varepsilon+b_k)^2}<$$

$$<\frac{\pi^3}{32(n+1)}\left(\frac{2a}{\varepsilon^2}+\int_{\varepsilon}^{\infty}\frac{dx}{x^2}\right)<\frac{\pi^3}{16\varepsilon(n+1)}.$$

Hence we get (8) if $C_1=\pi^3/16$.

It remains to show that (9) holds. By a simple transformation we get

(10) $$\int_{-\pi}^{\pi} P^2(x)\,dx=4\pi^2a^2\sum_{r=\varrho}^{\sigma}\sum_{s=\varrho}^{\sigma}\int_{-\pi}^{\pi} K_n(x-b_r)K_n(x-b_s)\,dx.$$

If $r\neq s$, for example $r<s$, then we can write

(11) $$\int_{-\pi}^{\pi} K_n(x-b_r)K_n(x-b_s)\,dx=\int_{-\pi}^{b_r-a}+\int_{b_r-a}^{b_r+a}+\int_{b_r+a}^{b_s-a}+\int_{b_s-a}^{b_s+a}+\int_{b_s+a}^{\pi}.$$

Let us denote the integrals on the right-hand side by I_1, I_2, I_3, I_4, I_5, respectively. Applying (4), (5) and (6), we get

$$(12)\quad I_1 \leqq \frac{\pi^2}{4(n+1)^2}\int_{-\pi}^{b_r-a}\frac{dx}{(x-b_r)^2(x-b_s)^2} < \frac{\pi^4}{4(n+1)^2(b_s-b_r)^2}\int_{-\pi}^{b_r-a}\frac{dx}{(x-b_s)^2} <$$

$$< \frac{\pi^2}{(s-r)^2(b_s-b_r+a)} < \frac{\pi^2}{(s-r)^2 a},$$

$$(13)\quad I_2 \leqq \frac{\pi^2}{2(n+1)}\,\frac{n+1}{2}\int_{b_r-a}^{b_r+a}\frac{dx}{(x-b_s)^2} < \frac{\pi^2}{4}\cdot\frac{2a}{(b_s-b_r-a)^2} =$$

$$= \frac{8a(n+1)^2}{(2s-2r-1)^2} = \frac{\pi^2}{2(2s-2r-1)^2 a},$$

and the same is true for I_5 and I_4 too, respectively. As to I_3, it is clear that $I_3=0$ if $s=r+1$. In case $s>r+1$ we break up the integral I_3 into the sum of the integrals J_1 and J_2 extended over $(b_r+a, (b_r+b_s)/2)$ and $((b_r+b_s)/2, b_s-a)$, respectively. J_1 may be estimated in the same way as I_1 and I_2, and we get

$$(14)\quad J_1 < \frac{4\pi^2}{(s-r)^2 a},$$

and the same is true for J_2.

In virtue of (11), (12), (13) and (14) we obtain that

$$\int_{-\pi}^{\pi} K_n(x-b_r)\,K_n(x-b_s)\,dx < 2(I_1+I_2+J_1) < \frac{11\pi^2}{(s-r)^2 a}.$$

Hence, using (6) and (10), it follows that

$$\int_{-\pi}^{\pi} P^2(x)\,dx < 4\pi^2 a^2 \sum_{r=\varrho}^{\sigma}\left(\pi n + \frac{22\pi^2}{a}\sum_{k=1}^{\infty}\frac{1}{k^2}\right) <$$

$$< 177\pi^4 a(\sigma-\varrho+1) = 177\pi^4\left(\frac{b_\sigma-b_\varrho}{2}+a\right) = C_2\delta,$$

if $C_2 = 177\pi^4$. This completes the proof of Lemma 1.

The following generalization of Lemma 1 can be proved by the same argument.

Lemma 1′. *Let $E\subset[-\pi/8, \pi/8]$ be a simple set, ε and $\eta(\leqq 1)$ positive real numbers, and n a natural number such that $n > C_1/\varepsilon\eta$. Then there exists a non-negative trigonometric polynomial $P(x)$ with frequencies 4ν $(\nu=0, 1, \ldots, n)$ such that*

$$(7')\quad P(x) \geqq 1 \quad \textit{if} \quad x\in E^{(\varepsilon)},$$

$$(8')\quad P(x) \leqq \eta \quad \textit{if} \quad x\in\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]-E,$$

and

$$(9')\quad \int_{-\pi}^{\pi} P^2(x)\,dx \leqq C_2\,\mathrm{mes}\,(E).^{2)}$$

[2]) mes (E) denotes the Lebesgue measure of the set E.

Lemma 2. *Let $P(x)$ be the trigonometric polynomial in Lemma 1′, and let N be a natural number divisible by 4, $N>4n+2$. Furthermore, set*

$$(15)\qquad \begin{aligned} Q_1(x) &= \cos Nx\cdot P(x),\\ Q_2(x) &= -C_3\cos 2x\cdot Q_1(x),\\ Q_3(x) &= C_4\cos 4Nx\cdot P(x). \end{aligned}$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies 2ν $(N/2-2n-1\leqq\nu\leqq 2N+2n)$, having the following properties:

$$(16)\qquad |Q_1(x)+Q_2(x)+Q_3(x)|\leqq C_5\eta \quad \textit{if}\quad x\in\left[-\frac{\pi}{8},\frac{\pi}{8}\right]-E,$$

$$(17)\qquad \int_{-\pi}^{\pi}(Q_1(x)+Q_2(x)+Q_3(x))^2\,dx\leqq C_6\,\mathrm{mes}\,(E).$$

Furthermore, there exists a decomposition of the set $E^{(\varepsilon)}$ into three simple, mutually disjoint subsets E_1, E_2, E_3, such that

$$(18)\qquad \sum_{k=1}^{l}Q_k(x)\geqq\frac{1}{4}\quad \textit{for}\quad x\in E_l\quad (l=1,2,3).$$

Proof. It is obvious that $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials, since $Q_1(x)$ and $Q_3(x)$ have only terms with frequencies divisible by 4, $Q_2(x)$ has only terms with frequencies divisible by 2, but not by 4, and, furthermore, we have $N+4n<4N-4n$.

In virtue of the fact that $\cos x\geqq 1/4$ if $|x|\leqq 5\pi/12$, we get the following estimates:

$$Q_1(x)\geqq\frac{P(x)}{4}\geqq\frac{1}{4}$$

if
$$x\in\bar E_1=E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(2k\pi-\frac{5\pi}{12}\right),\frac{1}{N}\left(2k\pi+\frac{5\pi}{12}\right)\right]\right\},$$

$$Q_1(x)+Q_2(x)\geqq-(C_3\cos 2x-1)\cos Nx\cdot P(x)\geqq\left(\frac{C_3}{4}-1\right)\frac{1}{4}$$

if
$$x\in\bar E_2=E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(2k\pi+\frac{7\pi}{12}\right),\frac{1}{N}\left(2k\pi+\frac{17\pi}{12}\right)\right]\right\},$$

and

$$Q_1(x)+Q_2(x)+Q_3(x)\geqq(C_4\cos 4Nx-C_3-1)P(x)\geqq\frac{C_4}{4}-C_3-1$$

if
$$x\in\bar E_3=E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(\frac{k\pi}{2}-\frac{5\pi}{48}\right),\frac{1}{N}\left(\frac{k\pi}{2}+\frac{5\pi}{48}\right)\right]\right\}.$$

Since $5\pi/12>\pi/2-5\pi/48$ and $7\pi/12<\pi/2+5\pi/48$, we have $E^{(\varepsilon)}=\bar E_1\cup\bar E_2\cup\bar E_3$.

Set $E_1=\bar{E}_1$, $E_2=\bar{E}_2-E_1$ and $E_3=\bar{E}_3-(E_1\cup E_2)$. We get (18) with $C_3=8$ and $C_4=4C_3+5$.

The inequalities (16) and (17) are then satisfied with $C_5=1+C_3+C_4$ and $C_6=1+C_3^2+C_4^2$. The proof of Lemma 2 is complete.

We shall need Lemma 2 in the following slightly different form too:

Lemma 2′. *Let $P(x)$ be an arbitrary trigonometric polynomial with even frequencies ν ($\leqq n$) and let N be an even natural number, $N>n+1$. Furthermore, set*

$$\text{(15′)}\qquad \begin{aligned} Q_1(x) &= \cos Nx\cdot P(x),\\ Q_2(x) &= -C_3\cos x\cdot Q_1(x),\\ Q_3(x) &= C_4\cos 4Nx\cdot P(x). \end{aligned}$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies ν $(N-n-1\leqq\nu\leqq 4N+n)$, having the following properties:

$$\text{(16′)}\qquad |Q_1(x)+Q_2(x)+Q_3(x)|\leqq C_5\,|P(x)|,$$

$$\text{(17′)}\qquad \int_{-\pi}^{\pi}(Q_1(x)+Q_2(x)+Q_3(x))^2\,dx\leqq C_6\int_{-\pi}^{\pi}P^2(x)\,dx.$$

Furthermore, every measurable set E $(\subset[-\pi/8,\pi/8])$, on which $P(x)$ is positive, can be decomposed into three mutually disjoint measurable subsets E_1, E_2, E_3, such that

$$\text{(18′)}\qquad \sum_{k=1}^{l}Q_k(x)\geqq\frac{P(x)}{4}\quad\textit{for}\quad x\in E_l\quad(l=1,2,3).$$

Lemma 3. *Let ε $(<\pi/4)$ be a positive real number. Then there exist mutually disjoint trigonometric polynomials $R_k^{(i)}(x)$ and simple sets $E_k^{(i)}$ $(k=1,2,\ldots,3^i;\,i=1,2,\ldots)$ with the following properties:*

(i) *the frequencies occurring in $R_k^{(i)}(x)$ $(k=1,2,\ldots,3^i)$ are even numbers, at most equal to a number $f_i=(C_7/\varepsilon)^i4^{4^i}$;*

$$\text{(ii)}\qquad \int_{-\pi}^{\pi}\Big(\sum_{k=1}^{3^i}R_k^{(i)}(x)\Big)^2dx\leqq C_8\quad\textit{for}\quad i=1,2,\ldots;$$

(iii) *the sets $E_k^{(i)}$ $(k=1,2,\ldots,3^i)$ corresponding to the same value of i are disjoint, the set*

$$F_i=\left[-\frac{\pi}{8},\frac{\pi}{8}\right]-\bigcup_{k=1}^{3^i}E_k^{(i)}$$

consists of at most $2f_i$ disjoint intervals, and

$$\text{(19)}\qquad \operatorname{mes}(F_i)\leqq\varepsilon\left(1-\frac{1}{2^i}\right);$$

(iv) *for any natural number i, the trigonometric polynomials $R_k^{(j)}(x)$ with $k=1,2,\ldots,3^j;\,j=1,2,\ldots,i$ can be arranged into a sequence*

$$U_1^{(i)}(x),\,U_2^{(i)}(x),\,\ldots,\,U_{J_i}^{(i)}(x)\quad\textit{where}\quad J_i=3+3^2+\ldots+3^i;$$

such that

$$\sum_{j=1}^{\mu_k^{(i)}} U_j^{(i)}(x) \geqq \frac{i}{8} \quad \textit{for every} \quad x \in E_k^{(i)} \tag{20}$$

with $\mu_k^{(i)}$ *not depending on the particular point* x *in* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$.

Remark to Lemma 3. On the basis of (i) and (ii), it is obvious that

$$\int_{-\pi}^{\pi} \Big(\sum_{j=1}^{J_i} U_j^{(i)}(x)\Big)^2 dx = \int_{-\pi}^{\pi} \Big(\sum_{j=1}^{i} \sum_{k=1}^{3^j} R_k^{(j)}(x)\Big)^2 dx \leqq C_8 i \tag{21}$$

holds for $i=1, 2, \ldots$.

Proof. The construction of the trigonometric polynomials $R_k^{(i)}(x)$ and sets $E_k^{(i)}$ will be accomplished by recurrence with respect to i.

First let $i=1$. Apply Lemma 1 with $\delta=\pi/8$, $\varepsilon/4$ instead of ε, $\eta=1$ and $n=[4C_1/\varepsilon]+1$.[3]) Then apply Lemma 2 for the obtained trigonometric polynomial and $N=4n+4$. We get the trigonometric polynomials $Q_k(x)$ and simple sets E_k $(k=1, 2, 3)$ satisfying (16), (17) and (18). Now write $R_k^{(1)}(x)=Q_k(x)$ and $E_k^{(1)}=E_k$ $(k=1, 2, 3)$. It is clear that $R_k^{(1)}(x)$ $(k=1, 2, 3)$ have even frequencies at most equal to

$$2(4N+4n) = 40n+32 \leqq \frac{C_7}{\varepsilon} 4^4 = f_1,$$

where $C_7=64\, C_1 C_5$. The assertions (ii) and (iii) are satisfied with $C_8=C_6\pi/4$, furthermore, the set F_1 consists of at most

$$2\sum_{k=1}^{3} \frac{4}{\pi} \operatorname{mes}(E_k^{(1)}) f_1 \leqq 2f_1$$

intervals. Writing $U_j^{(1)}(x)=R_j^{(1)}(x)$ and $\mu_j^{(1)}=j$ $(j=1,2,3)$, we have that (iv) holds to.o

Now we suppose that all the trigonometric polynomials $R_k^{(i)}(x)$ and sets $E_k^{(i)}$ with $i=1, 2, \ldots, m$ are already determined and satisfy (i)—(iv), and we are going to construct the polynomials and sets corresponding to $i=m+1$ so that the enlarged system still satisfy (i)—(iv).

We begin with applying Lemma 1′ by choosing subsequently $E_k^{(m)}$ $(k=1, 2, \ldots, 3^m)$ (instead of E), $\varkappa\varepsilon$ (instead of ε), η and $n>\max(C_1/\varkappa\varepsilon\eta, f_m)$, where the positive numbers $\varkappa, \eta$ and the natural number n will be determined later on. Denote by $P_k(x)$ $(k=1, 2, \ldots, 3^m)$ the corresponding trigonometric polynomials in the sense of Lemma 1′. Next apply Lemma 2 to each of the trigonometric polynomials $P_k(x)$ by choosing for the three functions (15) the following ones:

$$R_{3k-2}^{(m+1)}(x) = \cos N_k x \cdot P_k(x),$$

$$R_{3k-1}^{(m+1)}(x) = -C_3 \cos 2x \cdot R_{3k-2}^{(m+1)}(x),$$

$$R_{3k}^{(m+1)}(x) = C_4 \cos 4N_k x \cdot P_k(x)$$

[3]) The integer part of a real number α is denoted by $[\alpha]$.

($k=1, 2, \ldots, 3^m$), where the natural numbers N_k are chosen so that

$$\frac{N_1}{2}-2n-1>f_m, \quad \frac{N_{k+1}}{2}-2n-1>2N_k+2n$$

($k=1, 2, \ldots, 3^m-1$) and, in addition, each N_k be divisible by 4; we can choose for example:

(22) $$N_1=8n+4, \ N_{k+1}=4N_k+8n+4 \qquad (k=1, 2, \ldots, 3^m-1).$$

The condition (22) ensure that the trigonometric polynomials $R_k^{(m+1)}(x)$ ($k=1, 2, \ldots, 3^{m+1}$) are disjoint from one another and from all the polynomials $R_k^{(i)}(x)$ with $i \leqq m$. From (22) we get that the frequencies occurring in $R_k^{(m+1)}(x)$ are even numbers, at most equal to

(23) $$4N_{3^m}+4n<(1+4+4^2+\ldots+4^{3^m})(16n+1).$$

In virtue of (17) we get

$$\int_{-\pi}^{\pi}\Big(\sum_{k=1}^{3^{m+1}} R_k^{(m+1)}(x)\Big)^2 dx \leqq C_6 \sum_{k=1}^{3^m} \operatorname{mes}(E_k^{(m)}) \leqq C_8,$$

so that (ii) holds for $i=m+1$ too, with $C_8=C_6\pi/4$.

By Lemma 2 there exists a decomposition of the set $(E_k^{(m)})^{(\varkappa\varepsilon)}$ into three mutually disjoint simple subsets, which we denote now by $E_{3k-2}^{(m+1)}$, $E_{3k-1}^{(m+1)}$ and $E_{3k}^{(m+1)}$; thus

(24) $$\sum_{j=1}^{l} R_{3k-3+j}^{(m+1)}(x) \geqq \frac{1}{4} \quad \text{for} \quad x \in E_{3k-3+l}^{(m+1)}$$

($l=1, 2, 3$; $k=1, 2, \ldots, 3^m$). It is clear that the simple sets $E_k^{(m+1)}$ ($k=1, 2, \ldots, 3^{m+1}$) are disjoint. Using also the induction hypotheses, we get

$$\operatorname{mes}(F_{m+1}) \leqq \operatorname{mes}(F_m)+\operatorname{mes}\Big(\bigcup_{k=1}^{3^m}\big(E_k^{(m)}-(E_{3k-2}^{(m+1)}\cup E_{3k-1}^{(m+1)}\cup E_{3k}^{(m+1)})\big)\Big) \leqq$$

$$\leqq \varepsilon\Big(1-\frac{1}{2^m}\Big)+2f_m\cdot 2\varkappa\varepsilon.$$

Thus if we choose the hitherto indetermined $\varkappa$ such that

(25) $$\varkappa=\frac{1}{2^{m+3}f_m},$$

then (19) will be satisfied. We can easily see that F_{m+1} consists of at most $2f_{m+1}$ intervals, because

$$2\sum_{k=1}^{3^{m+1}} \frac{4}{\pi} \operatorname{mes}(E_k^{(m+1)}) f_{m+1} \leqq 2f_{m+1}.$$

This proved that (iii) holds.

The arrangement of trigonometric polynomials $R_k^{(i)}(x)$ with $k=1, 2, \ldots, 3^i$; $i=1, 2, \ldots, m, m+1$ into a sequence, as required by (iv), will be realized as follows. On the basis of induction hypothesis we have a sequence

$$U_1^{(m)}(x), U_2^{(m)}(x), \ldots, U_{J_m}^{(m)}(x) \tag{26}$$

of all the polynomials $R_k^{(i)}(x)$ with $i \leqq m$. For every trigonometric polynomial $R_k^{(m)}(x)$ $(k=1, 2, \ldots, 3^m)$ we find the place, where it occurs in the sequence (26), and then we insert the trigonometric polynomials

$$R_{3k-2}^{(m+1)}(x), \quad R_{3k-1}^{(m+1)}(x) \quad \text{and} \quad R_{3k}^{(m+1)}(x)$$

immediately after $R_k^{(m)}(x)$ in (26). In such a way we have ordered into a sequence $\{U_j^{(m+1)}(x)\}$ all the trigonometric polynomials $R_k^{(i)}(x)$ with $i \leqq m+1$. [4])

For every k $(k=1, 2, \ldots, 3^{m+1})$ let $\mu_k^{(m+1)}$ denote the subscrift j of that term of the sequence $\{U_j^{(m+1)}(x)\}$ which is equal to $R_k^{(m+1)}(x)$. A simple calculation shows that

$$\sum_{j=1}^{\mu_{3k-3+l}^{(m+1)}} U_j^{(m+1)}(x) \geqq \sum_{j=1}^{\mu_k^{(m)}} U_j^{(m)}(x) + \sum_{j=1}^{l} R_{3k-3+j}^{(m+1)}(x) - \sum_{j=1}^{3^{m+1}}{}' |R_j^{(m+1)}(x)|,$$

where the last sum is taken for each index j except $j=3k-3+l$ $(l=1, 2, 3)$. On the basis of the induction hypothesis, of (16) and (24), we get

$$\sum_{j=1}^{\mu_{3k-3+l}^{(m+1)}} U_j^{(m+1)}(x) \geqq \frac{m}{8} + \frac{1}{4} - (3^m - 1) C_5 \eta$$

for every $x \in E_{3k-3+l}^{(m+1)}$ $(l=1, 2, 3;\ k=1, 2, \ldots, 3^m)$, and this will be $\geqq (m+1)/8$ if we now fix the value of η as follows:

$$\eta = \frac{1}{8C_5(3^m-1)}. \tag{27}$$

This proved that (iv) holds for the case $m+1$ too.

Thus we have showed the properties (i)—(iv) with the exception, in (i), of the assertion concerning f_{m+1}, i.e. that $f_{m+1} = (C_7/\varepsilon)^{m+1} 4^{4^{m+1}}$. By (25) and (27), n must be chosen so that

$$n \geqq \max\left(\frac{C_1}{\varkappa \varepsilon \eta}, f_m\right) = \frac{64 C_1 C_5 \cdot 6^m f_m}{\varepsilon},$$

for example $n = [C_7 6^m f_m/\varepsilon] + 1$, where $C_7 = 64\, C_1 C_5$. By (23), the frequencies occurring in $R_k^{(m+1)}(x)$ equal at most

$$\frac{C_7 4^{3^m+2} 6^m f_m}{\varepsilon} < \frac{C_7^{m+1} 4^{4^{m+1}}}{\varepsilon^{m+1}} = f_{m+1}.$$

This completes the proof of Lemma 3.

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[4]) For example, in the case $m=1$ the sequence $\{U_j^{(2)}(x)\}$ will be the following: $R_1^{(1)}(x)$, $R_1^{(2)}(x)$, $R_2^{(2)}(x)$, $R_3^{(2)}(x)$, $R_2^{(1)}(x)$, $R_4^{(2)}(x)$, $R_5^{(2)}(x)$, $R_6^{(2)}(x)$, $R_3^{(1)}(x)$, $R_7^{(2)}(x)$, $R_8^{(2)}(x)$, $R_9^{(2)}(x)$.

Lemma 4. *Let M be an arbitrary natural number. Then for every m $(m=1, 2, \ldots)$ there exist mutually disjoint trigonometric polynomials $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ with the following properties:*

(v) *the frequencies ν occurring in $S_j^{(m)}(x)$ are such that $M+1 \leqq \nu \leqq 16M+4^{4^{m+C_9}}$;*

$$\text{(vi)} \qquad \int_{-\pi}^{\pi} \Big(\sum_{j=1}^{J_{m+1}} S_j^{(m)}(x)\Big)^2 dx \leqq \frac{C_{10}}{m} \quad (m=1, 2, \ldots);$$

$$\text{(vii)} \qquad \sum_{j=\mu_1}^{\mu_2} S_j^{(m)}(x) \geqq \frac{1}{32} \quad \textit{for every} \quad |x| \leqq \frac{\pi}{8},$$

where $\mu_i = \mu_i^{(m)}(x)$ $(i=1, 2)$, $1 \leqq \mu_1 \leqq \mu_2 \leqq J_{m+1}$ $(m=1, 2, \ldots)$.

Proof. Let us fix the natural number m. Apply Lemma 3 with $\varepsilon_1 = 1/m$. We get that there exist mutually disjoint trigonometric polynomials $U_j^{(m)}(x)$ $(j=1, 2, \ldots J_m)$, the frequencies occurring in $U_j^{(m)}(x)$ are even numbers, at most equal to the number f_m; furthermore, there exist disjoint simple sets $E_k^{(m)}$ $(k=1, 2, \ldots, 3^m)$ such that (iii), (20) and (21) hold.

Denote by F the simple set which can be obtained from the intervals $[\alpha, \beta]$ of $[-\pi/8, \pi/8] - \bigcup_{k=1}^{3^m} E_k^{(m)}$ by replacing them with $[\alpha-\varepsilon_2, \beta+\varepsilon_2]$, where $\varepsilon_2 = \varepsilon_1/4f_m$. It is clear that F consists of at most $2f_m$ intervals. In virtue of (iii), we have

$$\operatorname{mes}(F) \leqq \operatorname{mes}\left(\left[-\frac{\pi}{8}, \frac{\pi}{8}\right] - \bigcup_{k=1}^{3^m} E_k^{(m)}\right) + 4f_m\varepsilon_2 \leqq \frac{2}{m}.$$

Apply Lemma 1′ by choosing F (instead of E), ε_2 (instead of ε), $\eta=1$ and f_{m+1} (instead of n). We get the trigonometric polynomial $P^{(m)}(x)$ with frequencies 4ν $(\nu=0, 1, \ldots, f_{m+1})$ such that (7′) and (9′) hold.

Let N_1 and N_2 be the smallest even integers for which $N_1 - f_m \geqq M+1$ and $N_2 - 4f_{m+1} \geqq 4N_1 + f_m + 1$. Now apply Lemma 2′ to each of the trigonometric polynomials $U_{j_1}^{(m)}(x)/m$ with N_1, then to the trigonometric polynomial $P^{(m)}(x)/8$ with N_2 by choosing for the three functions (15′) the following ones:

$$S_j^{(m)}(x) = \cos N_1 x \cdot \frac{U_j^{(m)}(x)}{m},$$

$$S_{J_m+j}^{(m)}(x) = -C_3 \cos x \cdot S_j^{(m)}(x),$$

$$S_{2J_m+j}^{(m)}(x) = C_4 \cos 4N_1 x \cdot \frac{U_j^{(m)}(x)}{m}$$

$(j=1, 2, \ldots, J_m)$, furthermore,

$$S_{3J_m+1}^{(m)}(x) = \cos N_2 x \cdot \frac{P^{(m)}(x)}{8},$$

$$S_{3J_m+2}^{(m)}(x) = -C_3 \cos x \cdot S_{3J_m+1}^{(m)}(x),$$

$$S_{3J_m+3}^{(m)}(x) = C_4 \cos 4N_2 x \cdot \frac{P^{(m)}(x)}{8}.$$

By this the trigonometric polynomials $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ are defined because $3J_m+3=J_{m+1}$.

It is obvious that $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ are mutually disjoint trigonometric polynomials with frequencies at most equal to $4N_2+4f_{m+1}$. Now, a simple calculation shows

$$4N_2+4f_{m+1} \leqq 4(4N_1+4f_{m+1}+f_m+2)+4f_{m+1} \leqq 16M+20(f_{m+1}+f_m+2) <$$

$$< 16M+f_{m+2} = 16M+(C_7 m)^{m+2}4^{4^{m+2}} < 16M+4^{4^{m+C_9}}.$$

As to (vi), by (9′), (17′) and (21) we get that

$$\int_{-\pi}^{\pi}\Big(\sum_{j=1}^{J_{m+1}} S_j^{(m)}(x)\Big)^2 dx \leqq \frac{C_6}{m^2}\int_{-\pi}^{\pi}\Big(\sum_{j=1}^{J_m} U_j^{(m)}(x)\Big)^2 dx+$$

$$+\frac{C_2C_6}{64}\operatorname{mes}(F) \leqq \frac{C_{10}}{m}$$

holds with $C_{10}=C_6(C_8+C_2/32)$.

To show (vii), in case of $x\in E_k^{(m)}$ we set $\mu_1^{(m)}(x)=1$ and on the ground of (18′) $\mu_2^{(m)}(x)=\mu_k^{(m)}$ or $J_m+\mu_k^{(m)}$ or $2J_m+\mu_k^{(m)}$, respectively $(k=1, 2, \ldots, J_m)$. Furthermore, in case of $x\in F^{(\varepsilon_2)}$ we set $\mu_1^{(m)}(x)=3J_m+1$ and $\mu_2^{(m)}(x)=3J_m+1$ or $3J_m+2$ or $3J_m+3$ according to (18′). Thus the indices $\mu_1^{(m)}(x)$ and $\mu_2^{(m)}(x)$ $\big(1\leqq\mu_1^{(m)}(x)\leqq \leqq\mu_2^{(m)}(x)\leqq J_{m+1}\big)$ are defined for every $|x|\leqq\pi/8$ because

$$F^{(\varepsilon_2)} = \Big[-\frac{\pi}{8}, \frac{\pi}{8}\Big]-\bigcup_{k=1}^{3^m} E_k^{(m)}.$$

In virtue of (7′), (18) and (20), the assertion (vii) holds. So the proof is complete.

§ 2. Proof of the theorems

Without any loss of generality we may assume that $\{\varrho(n)\}$ is a non-decreasing sequence. Define the sequence of natural numbers $(C_9+1\leqq)\, m_1<m_2<\ldots$ such that

(28) $$\frac{\varrho(n)}{\sqrt{\log\log n}} \leqq \frac{1}{k} \quad \text{if} \quad n \geqq M_k = 4^{4^{m_k+C_9}}$$

$(k=1, 2, \ldots)$; this is possible by virtue of (1). Applying Lemma 4 with M_k, we get the trigonometric polynomials $S_j^{(m_k)}(x)$ $(j=1, 2, \ldots, J_{m_k+1};\ k=1, 2, \ldots)$. Denote by $T_k(x)$ the sum of the trigonometric polynomials $S_j^{(m_k)}(x-(k)_8\pi/4)$ [5] $(j=1, 2, \ldots, J_{m_k+1})$; it is obvious that

(29) $$T_k(x) = \sum_{n=M_k+1}^{17M_k} (a_n\cos nx+b_n\sin nx) \qquad (k=1,2,\ldots).$$

[5]) $(k)_8$ denotes the remainder of k modulo 8.

Consider the series

$$\text{(30)} \qquad \text{a)} \sum_{k=1}^{\infty} T_k(x), \qquad \text{b)} \sum_{k=1}^{\infty} \frac{\sqrt{m_k}}{k} T_k(x).$$

The trigonometric polynomials $T_k(x)$ and $T_l(x)$ do not overlap for $k \neq l$ because $17M_k \leqq M_k^4 \leqq M_{k+1}$ $(k=1, 2, \ldots)$. Therefore, writing every $T_k(x)$ in (30) in extenso, we represent (30) in the form of trigonometric series

$$\text{(31)} \qquad \text{a)} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad \text{b)} \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

where a_n and b_n are defined by (29), $A_n = a_n \sqrt{m_k}/k$ and $B_n = b_n \sqrt{m_k}/k$ if $M_k + 1 \leqq$ $\leqq n \leqq 17M_k$ $(k=1, 2, \ldots)$; and a_n, b_n, A_n, B_n equal 0 otherwise.

In virtue of (vi) and (28) the following estimates hold:

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \varrho^2(n) \leqq \sum_{k=1}^{\infty} \varrho^2(17M_k) \sum_{n=M_k+1}^{17M_k} (a_n^2 + b_n^2) \leqq$$

$$\leqq \sum_{k=1}^{\infty} \varrho^2(M_k^4) \int_{-\pi}^{\pi} T_k^2(x)\, dx \leqq C_{10} \sum_{k=1}^{\infty} \frac{m_k + C_9 + 1}{k^2 m_k} \leqq 2C_{10} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

and

$$\sum_{n=1}^{\infty} (A_n^2 + B_n^2) = \sum_{k=1}^{\infty} \frac{m_k}{k^2} \int_{-\pi}^{\pi} T_k^2(x)\, dx \leqq C_{10} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Hence, (31a) and (31b) are Fourier series of square integrable functions, and, in addition, (31a) satisfies (2).

Write down the mutually disjoint trigonometric polynomials $S_j^{(m_k)}(x)$ in this order:

$$\text{(32)} \quad S_1^{(m_1)}(x), \ldots, S_{J_{m_1}+1}^{(m_1)}(x); S_1^{(m_2)}(x), \ldots, S_{J_{m_2}+1}^{(m_2)}(x); \ldots; S_1^{(m_k)}(x), \ldots, S_{J_{m_k}+1}^{(m_k)}(x); \ldots$$

and label the occurring frequencies, in this order, by the subscript $n(j)$ $(j=1, 2, \ldots)$. It is clear that for the frequencies $n(j)$ occurring in the trigonometric polynomials $S_1^{(m_k)}(x), \ldots, S_{J_{m_k}+1}^{(m_k)}(x)$ of (32) $(k=1, 2, \ldots)$, we have

$$\text{(33)} \qquad M_k + 1 \leqq n(j) \leqq 17M_k.$$

It is obvious that series

$$\text{(34)} \qquad \begin{aligned} &\text{a)} \sum_{j=1}^{\infty} (a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x), \\ &\text{b)} \sum_{j=1}^{\infty} (A_{n(j)} \cos n(j)x + B_{n(j)} \sin n(j)x) \end{aligned}$$

are well deteinedrm arrangements of the non- vanishing terms of (31a) and (31b), respectively.

In virtue of (vii), the partial sums of (34a) diverge everywhere. Thus the proof of Theorem 1 is complete.

As to (3), denote by $\sigma_\lambda(x)$ the λ th partial sum of (34b). For any $x\in[-\pi/8, \pi/8]$ and for $k=1, 2, \ldots$ denote by $j_1=j_{1k}(x)$ the first natural number j, for which the frequency $n(j)$ occurs in $S_{\mu_1}^{(m_k)}(x)$, and by $j_2=j_{2k}(x)$ the last natural number j, for which the frequency $n(j)$ occurs in $S_{\mu_2}^{(m_k)}(x)$, where the subscripts $\mu_i=\mu_i^{(m_k)}(x)$ $(i=1, 2)$ are defined in Lemma 4, by (vii). Thus, we have

$$\sigma_{j_2}(x)-\sigma_{j_1}(x) = \frac{\sqrt{m_k}}{k}\sum_{j=\mu_1}^{\mu_2} S_j^{(m_k)}\left(x-(k)_8\frac{\pi}{4}\right),$$

and by (33), it is obvious that

$$M_k+1 \leqq j_{1k}(x) \leqq j_{2k}(x) \leqq 17M_k \qquad \left(k=1, 2, \ldots;\ |x| \leqq \frac{\pi}{8}\right).$$

Hence, for every $x\in[-\pi/8, \pi/8]$, $j_{1k}(x)$ and $j_{2k}(x)$ tend to ∞ with k. In virtue of (vii) and (28) we obtain

$$\frac{\sigma_{j_2}(x)-\sigma_{j_1}(x)}{\varrho(j_2)} \geqq \frac{\sqrt{m_k}}{32k\varrho(17M_k)} \geqq \frac{\sqrt{\log\log M_k^4}}{32\sqrt{2}\,k\varrho(M_k^4)} \geqq \frac{1}{32\sqrt{2}}$$

$(|x-(k)_8\pi/4| \leqq \pi/8;\quad k-1, 2, \ldots).$ Thus

$$\max\left(\frac{|\sigma_{j_2}(x)|}{\varrho(j_2)}, \frac{|\sigma_{j_1}(x)|}{\varrho(j_1)}\right) \geqq \frac{1}{64\sqrt{2}}.$$

Taking into account the construction of the trigonometric polynomials $T_k(x)$, (31b) satisfies (3) for every $x\in[-\pi, \pi]$. This concludes the proof of Theorem 2.

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