Smoothness conditions for Fourier series with monotone coefficients

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HARDY and LITTLEWOOD [2] showed that for Fourier series with monotone coefficients it is possible to connect the integrability of the function and the summability of the coefficients. We show how it is possible to get a similar theorem connecting the smoothness of the function with the summability of the coefficients.

Theorem 1. *Let* $0<\alpha<2$, $1<p<\infty$, $1\leqq q\leqq\infty$,

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx,\quad a_{n+1}\leqq a_n.$$

Then

$$\left[\sum a_n^q n^{q(\alpha+1-1/p)-1}\right]^{1/q}$$

is finite if and only if

$$\left[\int_0^{\pi}\left[\int_0^{\pi}\left|\frac{f(x+t)-2f(x)+f(x-t)}{t^{\alpha}}\right|^p dx\right]^{q/p}\frac{dt}{t}\right]^{1/q} \tag{1}$$

is finite.

The class of functions for which (1) is finite is usually denoted by $\Lambda(\alpha,p,q)$ and has been extensively studied by TAIBLESON [8]. Some special cases of this theorem have been found previously; $p=q=\infty$, $0<\alpha<1$ in [7], $1<p<\infty$, $q=\infty$, $0<\alpha<1$ in [4], and a number of different cases in [6]. To simplify the exposition of this note we use two results that are implicity contained in [1]:

$$a_n\leqq An^{-1+1/p}\inf_{\frac{\pi}{(n+1)}\leqq t\leqq\frac{\pi}{n}}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{1/p}, \tag{2}$$

$$\left\{\begin{aligned}&\sup_{0\leqq t\leqq\pi/n}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{1/p}\leqq\\&\leqq An^{-2}\left[\sum_{k=1}^{n}k^{3p-2}a_k^p\right]^{1/p}+B\left[\sum_{k=n}^{\infty}k^{p-2}a_k^p\right]^{1/p};\end{aligned}\right. \tag{3}$$

(2) is also contained in [5].

[1]) Supported in part by N. S. F. grant GP-3483.

Finally we need a form of HARDY's inequality and also the reverse inequality to HARDY's inequality which holds when the terms are monotone.

Theorem A. *If* (a) $c>1$, $s_n=a_1+\dots+a_n$, *or* (b) $c<1$, $s_n=a_n+a_{n+1}+\dots$, *then*

$$\sum n^{-c}s_n^p \leqq K\sum n^{-c}(na_n)^p, \qquad 1<p<\infty.$$

Theorem B. *If* (a) $c>1$, $s_n=a_1+\dots+a_n$, *or* (b) $c<1$, $s_n=a_n+a_{n+1}+\dots$, *and* $n^{-k}a_n$ *is monotone for some* k, *then*

$$\sum n^{-c}s_n^p \leqq K\sum n^{-c}(na_n)^p, \qquad 0<p<1.$$

Theorem A is in [3, p. 255] and Theorem B is in [5, p. 75 and p. 83].

Assume first that (1) is finite. By (2) we have

$$n^{(\alpha+1-1/p)q-1}a_n^q \leqq A \inf_{\frac{\pi}{n+1}\leqq t\leqq\frac{\pi}{n}} \left[\int_0^\pi |f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p} n^{\alpha q-1} \leqq$$

$$\leqq A\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\left[\int_0^\pi |f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}$$

and summing we have

$$\sum n^{(\alpha+1-1/p)q-1}a_n^q \leqq A\int_0^\pi\left[\int_0^\pi |f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}.$$

To prove the other inequality we use (3) and Theorems A and B. By (3)

$$\int_0^\pi\left[\int_0^\pi |f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}=$$

$$=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\left[\int_0^\pi |f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}\leqq$$

$$\leqq A\sum_{n=1}^{\infty} n^{\alpha q-1}n^{-2q}\left[\sum_{k=1}^{n}k^{3p-2}a_k^p\right]^{q/p}+A\sum_{n=1}^{\infty}n^{\alpha q-1}\left[\sum_{k=n}^{\infty}k^{p-2}a_k^p\right]^{q/p}.$$

If $q/p<1$ we use Theorem B. If $q/p>1$ we use Theorem A and if $q=p$ we interchange the order of summation. Then we get

$$(1)\leqq A\sum_{n=1}^{\infty}n^{\alpha q-1-2q}[n^{3p-1}a_n^p]^{q/p}+A\sum_{n=1}^{\infty}n^{\alpha q-1}[n^{p-1}a_n^p]^{q/p}=A\sum_{n=1}^{\infty}n^{(\alpha+1-1/p)q-1}a_n^q.$$

The conditions on the parameters that must be satisfied to use Theorems A and B are all implied by the condition $0<\alpha<2$. The proof for the case $q=\infty$ is an obvious adaptation of the above proof.

Bibliography

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(Received July 2, 1966)