On the index of imprimitivity of a non-negative matrix

By ŠTEFAN SCHWARZ in Bratislava (ČSSR)

1.

Let A be a non-negative $n \times n$ matrix. To study the distribution of zeros and non-zeros in the matrices of the sequence

$$A, A^2, A^3, \ldots \tag{1}$$

we have introduced in [2] the following notations. Consider the set of symbols $E = \{e_{ij} | i, j = 1, 2, \ldots, n\}$ together with a zero 0 adjoined. Define in $S = \{0\} \cup E$ a multiplication by

$$e_{ij} e_{lm} = \begin{cases} e_{im} & \text{for } j = l, \\ 0 & \text{for } j \neq l, \end{cases}$$

the zero element having the usual properties of a multiplicative zero. Then S (with this multiplication) is a semigroup.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support C_A of A we shall mean the subset of S containing 0 and all e_{ij} for which $a_{ij} > 0$.

For two non-negative $n \times n$ matrices A, B we have $C_{AB} = C_A C_B$, where the product to the right has the usual meaning used in the theory of semigroups.

In particular the supports of the elements of the sequence (1) are

$$C_A, C_A^2, C_A^3, \ldots. \tag{2}$$

Since this sequence has only a finite number of different elements (subsets of S) it can be written in the form

$$C_A, C_A^2, \ldots, C_A^{k-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid \ldots.$$

Here C_A^k, $k = k(A)$, is the least power in (2) which appears more than once and d is the period with which all the following powers repeat.

Denote further $S_i = \{0\} \cup \{e_{i1}, e_{i2}, \ldots e_{in}\}$ and $F_i = F_i(A) = S_i \cap C_A$, so that F_i is the "support" of the i-th row in A.

The sequence

$$F_i, F_i C_A, F_i C_A^2, \ldots$$

contains again only a finite number of different elements (subsets of S_i) and it is of the form

$$F_i, F_i C_A, \ldots, F_i C_A^{k_i-2} \mid F_i C_A^{k_i-1}, \ldots, F_i C_A^{k_i+d_i-2} \mid F_i C_A^{k_i-1}, \ldots$$

where the integers k_i, d_i have an analogous meaning as the integers k and d above.

For details concerning these notions see [3].

In [3] and [4] we have proved:

Lemma 1. *For any non-negative* $n\times n$ *matrix* A *we have:*

a) $k(A)=\max(k_1, k_2, \dots, k_n)$;

b) $d(A)=\text{l.c.m.}\,[d_1, d_2, \dots, d_n]$.

Lemma 2. *If* A *is irreducible, then* $d(A)=d_1=d_2=\dots=d_n$.

Denote by g_i the number of non-zero elements in F_i. In the papers [3] and [4] we have found some estimates concerning the numbers k_i in terms of n and g_i. For instance we have proved $k_i\leqq 1+(n-g_i)(n-g_i+1)$.

It is intuitively clear that also the numbers d_i depend on g_i. It is the purpose of this paper to give an estimation concerning $d=d(A)$ in terms of n and g_i. For an irreducible matrix A the number d is identical with the classical notion of the index of imprimitivity of A. Our main result is formulated in the theorem below.

2.

Let M be any non-negative $n\times n$ matrix. It is well known that there is a permutation matrix P such that PMP^{-1} is of the form

$$A=\begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & & & \\ A_{r1} & A_{r2} & \dots & A_{rr}\end{pmatrix},$$

where $A_{\alpha\alpha}$ are irreducible matrices (including the case that some of the $A_{\alpha\alpha}$ are zero matrices of order 1). It is easy to see that $d(A)=d(M)$. Further it can be proved (see [1], [5]) that $d(A)=\text{l.c.m.}\,[d(A_{11}), d(A_{22}), \dots, d(A_{rr})]$. Hence $d(A)$ does not depend on the rectangular matrices $A_{\alpha\beta}$, $\alpha\neq\beta$.

It is therefore sufficient to restrict ourselves to the case of an *irreducible* matrix A.

In [3] we have proved:

Lemma 3. *If* A *is irreducible (of order* n*), then there is an integer* h_i *such that* $1\leqq h_i\leqq n$ *and* $F_i\subset F_iC_A^{h_i}$. *Here:*

a) *if* $e_{ii}\in F_i$, *we may choose* $h_i=1$,

b) *if* F_i *contains* g_i *non-zero elements* $\in S_i$, *we have, for the least number* h_i *satisfying the above condition,* $h_i\leqq n-g_i+1$.

Consider now the chain

$$F_i\subset F_iC_A^{h_i}\subset F_iC_A^{2h_i}\subset\dots$$

Since any member of this chain contains at most $n+1$ different elements (namely the elements $0, e_{i1}, \dots, e_{in}$) there is an integer $\tau\geqq 1$ such that

$$F_iC_A^{\tau h_i}=F_iC_A^{\tau h_i+h_i}, \tag{3}$$

hence $d_i \leqq h_i \leqq n - g_i + 1$. With respect to the definition of the number d_i we conclude from (3) that $d_i | h_i$. By Lemma 2 we obtain $d | h_i$ for $i = 1, 2, \ldots, n$.

We have proved:

Lemma 4. *Let A be irreducible. Denote $\delta = (h_1, h_2, \ldots, h_n)$. We then have $d | \delta$.*

Lemma 4 implies $d \leqq \delta = (h_1, \ldots, h_n) \leqq \min_i h_i \leqq n + 1 - \max_i g_i$. We have proved:

Theorem. *Let A be an irreducible non-negative $n \times n$ matrix. Denote by g_i the number of positive entries in the i-th row of A. Then $d(A) \leqq n + 1 - \max_i g_i$.*

Remark. In terms of the integers h_i we may state the following. If $d < \min_i h_i$, then $d | \min_i h_i$ implies that we certainly have $d \leqq \frac{1}{2} \min_i h_i$. If here again the equality does not hold, we have $d \leqq \frac{1}{3} \min_i h_i$. And so on.

3.

We now give some corollaries.

Suppose that $d(A) = n$. Then $n \leqq n + 1 - \max_i g_i$ implies $g_i = 1$ for $i = 1, 2, \ldots, n$. Hence C_A is the support of an (irreducible) permutation matrix. This can be stated in the following forms:

Corollary 1. *Suppose that for an irreducible non-negative $n \times n$ matrix A we have $d(A) = n$. Then the matrix obtained by replacing the positive entries in A by the number 1 is a permutation matrix.*

Corollary 2. *Let A be a non-negative irreducible $n \times n$ matrix. Suppose that replacing all positive entries in A by the number 1 we obtain a matrix which is not a permutation matrix. Then $d(A) \leqq n - 1$.*

Example 1. In general the result of Corollary 2 cannot be sharpened. This is shown on the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Here

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and $C_A = C_A^4$. Therefore $d(A) = 3$. In this case $h_i = 3$ $(i = 1, 2, 3, 4)$ and $d(A) = \min_i h_i$.

Example 2. If $\max_i g_i = n - 1$, our Theorem implies $d(A) \leqq 2$. This result is sharp in the following sense. To any $n \geqq 2$ there is an irreducible $n \times n$ matrix A

with $\max_i g_i = n-1$ such that $d(A)=2$. This property has for instance the matrix

$$A = \begin{pmatrix} 0 & 1 \dots 1 \\ 1 & 0 \dots 0 \\ \vdots & \\ 1 & 0 \dots 0 \end{pmatrix}.$$

Here

$$A^2 = \begin{pmatrix} n-1 & 0 \dots 0 \\ 0 & 1 \dots 1 \\ \vdots & \\ 0 & 1 \dots 1 \end{pmatrix}.$$

Since $C_A \cup C_A^2 = S$, the matrix A is irreducible (see [2], Theorem 1) and clearly we have $d(A)=2$.

The result of our Theorem is also sharp in the following sense. To any n and any g, $1 \leqq g \leqq n-1$, there exist numbers $g_1, \dots, g_n$ with $\max(g_1, \dots, g_n) = g$ and a matrix A having g_i non-zero elements in the ith row of A such that $d(A) = n+1-g$. Take for this purpose the matrix A with $C_A = \{0, e_{12}, e_{23}, \dots, e_{n-g, n-g+1}, e_{n-g+1,1}, e_{1,n-g+2}, \dots, e_{1,n}, e_{n-g+2,3}, \dots, e_{n,3}\}$. Here $g_1 = g, g_2 = \dots = g_n = 1$. It can be shown that $C_A = C_A^{n-g+2}$ and $n-g+2$ is the least number $l \neq 1$ satisfying $C_A^l = C_A$. Hence $k(A)=1$ and $d(A) = n-g+1$.

If at least one of the numbers h_i is equal to 1, we have $d=\delta=1$. This means that some power of A is positive. Such a matrix is called *primitive*. Hence:

Corollary 3. *If an irreducible matrix A contains at least one row with $F_i \subset F_i C_A$, then A is primitive.*

By Lemma 3 this is certainly the case if $e_{ii} \in F_i$ for some i. This implies the following well-known result which goes back to Frobenius:

Corollary 4. *If A is irreducible and it contains a positive entry in the main diagonal, then A is primitive.*

Remark. The condition $F_i \subset F_i C_A$ is weaker then the condition $e_{ii} \in F_i$. For instance, for a matrix A with

$$C_A = \begin{Bmatrix} e_{11} & e_{12} & 0 \\ 0 & 0 & e_{23} \\ e_{31} & 0 & e_{33} \end{Bmatrix}$$

we have $F_2 = \{0, e_{23}\} \subset F_2 C_A = \{0, e_{21}, e_{23}\}$, while $e_{22} \notin F_2$.

Since $d \mid \delta = (h_1, \dots, h_n)$ we also have:

Corollary 5. *If two of the numbers h_i are relatively prime, then A is primitive.*

Example 3. The following example shows that $d<\delta$ is possible. Consider the matrix A and its powers:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

For $i=1,2,3$ we have $F_i \not\subset F_i C_A$ but $F_i \subset F_i C_A^2$, so that $h_1=h_2=h_3=2$; hence $\delta=2$. But our matrix is primitive, i.e., $d(A)=1<\delta$.

Remark. It is worth to remark that the set $\{h_i\}$ is not identical with an other set of integers (denoted below by $\{r_i\}$), which can be associated to any irreducible (and some reducible) non-negative matrices. Let A be irreducible. Denote by r_i the least integer $\geqq 1$ such that $e_{ii} \in F_i C_A^{r_i-1}$ and define $F_i C_A^0 = F_i$. For an irreducible matrix r_i always exists and we have $r_i \leqq n$. (In the graph-theoretical treatment of non-negative matrices the r_i's are the lengths of elementary circuits.) Since $e_{ii} \in F_i C_A^{r_i-1}$ implies $F_i = e_{ii} C_A \subset F_i C_A^{r_i}$, we have $h_i \leqq r_i \leqq n$. It is known that $d=(r_1, r_2, \ldots, r_n)$ in contradistinction to $d \leqq (h_1, h_2, \ldots, h_n)$.

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