Norm relations and skew dilations

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An operator T on a Hilbert space H possesses a skew dilation if there exists a Hilbert space $K \supset H$, a constant $\varrho > 0$, and a unitary operator U on K, such that $T^n = \varrho P U^n P$ for $n = 1, 2, \ldots$, where P is the self-adjoint projection of K on H. If $T^n = \varrho P U^n P$, then following the notation of [5], we say $T \in C_\varrho$. Note that C_1 is the class of all contractions and C_2 is the set of all operators with numerical range in the unit disc. SZ.-NAGY and FOIAŞ [5] have characterized C_ϱ for general $\varrho > 0$.

In the first part of this paper, we obtain bounds on $\|T^n x\|$ for $T \in C_\varrho$. These bounds should be useful in constructing a matrix dilation for $T \in C_\varrho$ similar to the Schäffer dilation for contractions. The rest of the paper is devoted to general results on C_ϱ.

It is convenient to write $T^n = \varrho P U^n P$ or $\delta T^n = P U^n P$ depending on the context. For the rest of the paper it is assumed that $\delta = \varrho^{-1}$.

Lemma 1. *Let $\delta T^j = P U^j P$ for $j = 1, 2, \ldots$. Then $P U^k [(I-P)U]^n P = \alpha_n T^{n+k}$ for $k, n = 1, 2, \ldots$, where α_n is independent of k.*

Proof. Expand $[(I-P)U]^n P$ formally. Then all terms are either of the form aT^n or $bU^j T^{n-j}$, after simplifications via the relation $PU^m P = \delta T^m$. But $PU^k a T^n = \delta a T^{n+k}$ and $PU^k b U^j T^{n-j} = \delta b T^{n+k}$. Thus, $PU^k[(I-P)U]^n P = \sum_m c_m T^{n+k}$, where the constants c_m do not depend on k, but only upon the coefficients arising in the formal expansion of $[(I-P)U]^n P$, and the subsequent conversion of U's to T's.

Corollary. *Let $\delta T^j = P U^j P$ for $j = 1, 2, \ldots$. Then $PU^k[(I-P)U]^n U^m P = \alpha_n T^{k+n+m}$, where α_n is independent of k and m.*

Proof. Same as above.

Theorem 1. $PU[(I-P)U]^n P = \delta(1-\delta)^n T^{n+1}$.

Proof. We assume that the relation is true for n and check it for $n+1$. (It obviously holds for $n = 1$.)

$$PU[(I-P)U]^{n+1}P = PU^2[(I-P)U]^n P - PU\{PU[(I-P)U]^n P\} =$$
$$= \delta(1-\delta^n)T^{n+2} - \delta T[\delta(1-\delta)^n T^{n+1}] = \delta(1-\delta)^{n+1} T^{n+2}.$$

To convert the first term on the right, we have made use of Lemma 1 and the induction hypothesis.

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Corollary. *Let* $\delta T^j = PU^jP$ *for* $j=1, 2, \ldots$. *Then* $PU^k[(I-P)U]^n U^m P = \delta(1-\delta)^n T^{n+k+m}$ *for* $n, k=1, 2, \ldots$.

Proof. Same as above using the Corollary to Lemma 1.

Lemma 2. *Let* V *be an isometry and* P *a self-adjoint projection on a Hilbert space. Then*

$$\|x\|^2 = \sum_{n=0}^{M-1} \|PV[I-P)V]^n x\|^2 + \|[(I-P)V]^M x\|^2$$

for $M=1, 2, \ldots$.

Proof. By induction.

Corollary. *Under the same hypothesis as above,*

$$\|x\|^2 = \sum_{n=0}^{M-1} \|PV[(I-P)V]^n V^k x\|^2 + \|[(I-P)V]^M V^k x\|^2$$

for $M, k=1, 2, \ldots$.

Proof. Replace x by $V^k x$ in Lemma 2.

Theorem 2. *Let* $\delta T^j = PU^jP$ *for* $j=1, 2, \ldots$. *Then*

$$\sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)} \|T^n x\|^2 \leqq \|x\|^2.$$

Proof. For M fixed, it follows from Lemma 2 and Theorem 1 that,

$$\|x\|^2 \geqq \sum_{n=0}^{M-1} \|PU[(I-P)U]^n x\|^2 = \sum_{n=1}^{M} \|\delta(1-\delta)^{n-1} T^n x\|^2.$$

Letting $M \to \infty$ completes the proof.

Corollary 1. *Let* $\delta T^j = PU^jP$ *for* $j=1, 2, \ldots$. *Then*

$$\sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)} \|T^{n+k} x\|^2 \leqq \|x\|^2 \quad \textit{for} \quad k=1, 2, \ldots.$$

Proof. Same as above, using the Corollaries to Lemma 2 and Theorem 1.

Corollary 2. *If* $|W(T)| \leqq 1$, *then*

$$\sum_{n=1}^{\infty} 4^{-n} \|T^{n+k} x\|^2 \leqq \|x\|^2 \quad \textit{for} \quad k=1, 2, \ldots.$$

Proof. By [1], we know that $|W(T)| \leqq 1$ implies $\frac{1}{2}T^j = PU^jP$ for $j=1, 2, \ldots$.

Corollary 3. *Let* $\delta T^j = PU^jP$ *for* $j=1, 2, \ldots$. *If*

$$\sum_{n=1}^{M} \delta^2(1-\delta)^{2(n-1)} \|T^n x\|^2 = \|x\|^2, \quad \textit{then} \quad T^{M+1} x = 0.$$

Corollary 4. *Let* $|W(T)|\leqq 1$. *If*

$$\sum_{n=1}^{M} 4^{-n}\|T^n x\|^2 = \|x\|^2, \quad \text{then} \quad T^{M+1}x = 0.$$

Note that Corollary 4 gives us a much sharper form of the following result from [2]: If $|W(T)|\leqq 1$ and $\|Tx\|=2\|x\|$, then $T^2x=0$.

Theorem 3. *Let* $T^j=\varrho PU^jP$ *for* $j=1, 2, \ldots$ *where* $\varrho>1$. *If* $\liminf\|T^n x_0\| = \alpha\|x_0\|$, *then* $\alpha\leqq(2\varrho-1)^{1/2}$.

Proof. Assume $\alpha>(2\varrho-1)^{1/2}$ for $x_0\in H$. Then for some fixed k, $\|T^{n+k}x_0\| > (2\varrho-1)^{1/2}\|x_0\|$ for $n=1, 2, \ldots$. Thus,

$$\|x_0\|^2 \geqq \sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)}\|T^{n+k}x_0\|^2 >$$

$$> (2\varrho-1)\|x_0\|^2 \sum_{n=0}^{\infty} \delta^2(1-\delta)^{2n} = (2\varrho-1)\delta^2\|x_0\|^2/(2\delta-\delta^2) = \|x_0\|^2,$$

which is impossible.

Corollary. *If* $|W(T)|\leqq 1$ *and* $\liminf\|T^n x_0\|=\alpha\|x_0\|$, *then* $\alpha\leqq\sqrt{3}$.

Proof. For $|W(T)|\leqq 1$, it follows from [1] that $T^j=\varrho PU^jP$, where $\varrho=2$.

In [2] an example was given of an operator T where $|W(T)\leqq 1$ and $\lim\|T^n x_0\| = \sqrt{2}\|x_0\|$. This raises the question of the best possible constant K, such that a) $|W(T)|\leqq 1$ and b) $\liminf\|T^n x_0\|\leqq K\|x_0\|$. The Corollary to Theorem 3 does not give a sharp answer to this question, but it does reduce the upper bound on K from 2 to $\sqrt{3}$.

Note that if $\varrho<1$, then $\|T^n x_0\|\leqq\varrho^n\|x_0\|\to 0$. However, it is still possible to obtain a weak form of Theorem 3.

Theorem 4. *Let* $T^j=\varrho PU^jP$ *for* $j=1, 2, \ldots$, *where* $\varrho<1$. *If* $\|T^n x_0\|\geqq\alpha\varrho^n\|x_0\|$ *for all* n, *then* $\alpha\leqq[\varrho(2-\varrho)]^{1/2}$.

Proof. Assume $\alpha>[\varrho(2-\varrho)]^{1/2}$ for $x_0\in H$. Then

$$\|x_0\|^2 \geqq \sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)}\|T^n x_0\|^2 \geqq$$

$$\geqq \alpha^2\|x_0\|^2 \sum_{n=0}^{\infty} (1-\delta)^{2n}\varrho^{2n} > \varrho(2-\varrho)\|x_0\|^2 \sum_{n=0}^{\infty} (1-\varrho)^{2n} = \|x_0\|^2$$

which is impossible.

Lemma 3. *Let* $T^j=\varrho PU^jP$ *for* $j=1, 2, \ldots$, *and let* $f(z)$ *be analytic for* $|z|<1$ *and continuous on the boundary. Then* $\lim_{r\to 1-} f(rT)$ *exists, and equals* $(1-\varrho)f(0)I + \varrho Pf(U)P$, *where convergence is in the norm topology.*

Proof. Since $\|T^j\|\leqq\varrho$ for $j=1, 2, \ldots$, it is clear that $f(rT)=\sum_0^\infty a_n r^n T^n$ converges absolutely, for $r<1$. Indeed, $\|\sum_0^\infty a_n r^n T^n\|\leqq\sum_0^\infty |a_n| r^n\varrho$ and $|a_n|\leqq M$ since

f is continuous. Thus,

$$f(rT) = \sum_{n=0}^{\infty} a_n r^n T^n = a_0 I + \varrho P \sum_{n=1}^{\infty} a_n r^n U^n P =$$

$$= (1-\varrho) a_0 I + \varrho P \sum_{0}^{\infty} a_n r^n \int e^{int}\, dE(t) P =$$

$$= (1-\varrho) a_0 I + \varrho P \int \Big[\sum_{0}^{\infty} a_n r^n e^{int}\Big]\, dE(t) P =$$

$$= (1-\varrho) a_0 I + \varrho P \int f(re^{it})\, dE(t) P, \quad \text{for} \quad r \leqq R < 1.$$

Since $f(z)$ is continuous in $|z| \leqq 1$, it follows that

$$\lim_{r\to 1-} f(rT) = (1-\varrho) a_0 I + \varrho P \int f(e^{it})\, dE(t) P = (1-\varrho) f(0) I + \varrho\, Pf(U) P.$$

Theorem 5. *Let $T \in C_\varrho$. Let $f(z)$ be analytic in $|z| < 1$ and continuous on the boundary, where $f(0) = 0$ and $|f(z)| \leqq 1$ for $|z| \leqq 1$. Then $f(T) \in C_\varrho$.*

Proof. Let $g(z) = [f(z)]^n$. Then it follows from Lemma 3 that $[f(T)]^n = g(T) = \varrho P g(U) P = \varrho P [f(U)]^n P$ for $n = 1, 2, \dots$. Since U is unitary, it follows that, while $f(U)$ is not necessarily unitary, it is a contraction. Hence $f(U)$ has a unitary dilation, which completes the proof.

This theorem appeared in [5] under the additional assumption that $f(z)$ have an absolutely convergent Taylor series.

A little thought about Theorem 4 reveals that if T is normal and $\|T\| = \varrho < 1$, then $T \notin C_\varrho$. This leads one to ask how large a normal operator can be and still be a successful candidate for membership in C_ϱ.

While preparing the manuscript, we learned that this question had been answered independently by E. Durszt [6]. Our results are slightly more general, and for that reason we include the statements of Lemmas 4 and 5 and Theorem 6. Since the proofs are implicitly contained in Durszt's paper, we omit them. (The observation that all points in the boundary of the spectrum of an operator lie in the approximate point spectrum is relevant to Lemma 5.)

Lemma 4. *If $\|T\| \leqq \varrho/(2-\varrho)$ and $\varrho < 1$, then $T \in C_\varrho$. If $\|T\| \leqq 1$, then $T \in C_\varrho$ for $\varrho \geqq 1$.*

Lemma 5. *If $T \in C_\varrho$ for $\varrho < 1$, then $R_{sp}(T) \leqq \varrho/(2-\varrho)$. If $T \in C_\varrho$ for $\varrho \geqq 1$, then $R_{sp}(T) \leqq 1$.*

Theorem 6. *Let T be normaloid. For $\varrho \leqq 1$, $T \in C_\varrho$ if and only if $\|T\| \leqq \varrho/(2-\varrho)$. For $\varrho \geqq 1$, $T \in C_\varrho$ if and only if $\|T\| \leqq 1$.*

Note that hyponormal, subnormal, normal, self adjoint and unitary operators are all normaloid.

In [5], there is an example of a power bounded operator which is not in C_ϱ for any ϱ. We will now present a simpler example which does slightly more than theirs.

First we need the following

Theorem B. (SZ.-NAGY—FOIAŞ) *For $\varrho>2$, $T\in C_\varrho$ if and only if*
1) $\sigma(T)\subset\{z: |z|\leqq 1\}$,
2) $\|(zI-T)^{-1}\|\leqq(|z|-1)^{-1}$ *for* $1<|z|\leqq(\varrho-1)/(\varrho-2)$.

Theorem 7. *Given $a>0$, there exists an operator T such that*
1) $\|T^n\|\leqq 1+a$ *for* $n=1, 2, \ldots$
2) $T\notin C_\varrho$ *for any* ϱ.

Proof. Given $a>0$, our operator T is defined as follows: $T\varphi_1=\varphi_1+a\varphi_2$, $T\varphi_2=-\varphi_2$, where $\{\varphi_1, \varphi_2\}$ is an orthonormal basis for the space H. Since $T^2=T$, it is clear that $\|T^n\|\leqq 1+a$ for $n=1, 2, \ldots$. However,

$$(zI-T)^{-1}\varphi_1=(z-1)^{-1}[\varphi_1+a(1+z)^{-1}\varphi_2] \quad \text{for} \quad z\neq\pm 1.$$

Since $\|(zI-T)^{-1}\varphi_1\|=|z-1|^{-1}[1+a^2/|1+z|^2]^{1/2}$, T does not satisfy condition 2 of Theorem B for any $\varrho>2$; as may be seen by taking z real with $1<z\leqq\leqq(\varrho-1)(\varrho-2)^{-1}$. However, $C_\alpha\subset C_\beta$ for $\alpha<\beta$ (see [5]) which implies $T\notin C_\varrho$ for any $\varrho>0$ as promised.

Added in proof: Recently we received a preprint "Remarks on the numerical radius" from TOSIO KATO. Combining an idea from that paper with the existing results and techniques of this one, it is possible to obtain a remarkable sharpening of Theorem 2 and its Corollaries.

Theorem 2′. *Let $\{k_n\}$ be any strictly increasing sequence of positive integers. Let $\delta T^j=PU^jP$. Then*

$$\sum_{n=1}^{\infty}\delta^2(1-\delta)^{2(n-1)}\|T^{k_n}x\|^2\leqq\|x\|^2.$$

The proof involves a fairly simple modification of the argument with particular emphasis on Lemma 2.

In Theorem 3, lim inf may now be replaced by lim sup, and we obtain the following:

Corollary. *If $|W(T)|\leqq 1$, then* $\limsup\|T^nx\|\leqq\sqrt{3}\|x\|$.

This sharpens KATO's bound of $4/\sqrt{5}$ for this lim sup.

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