On a process concerning inaccessible cardinals. III

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This paper is a continuation of references I and II (see [1] and [2]), in which a process concerning inaccessible cardinals has been defined. In this paper we freely make use of the notations and theorems of [2].

In [1] the process was described by a sequence of functions

$$f_0(\alpha^{(0)}),\quad f_1(\alpha^{(0)}, \alpha^{(1)}), \ldots, f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)}), \ldots$$

where the variables η and $\alpha^{(\eta)}$ run over all ordinal numbers and for given η the functions $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ are defined for such arguments $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ in which only a finite number of terms is distinct from 0.

In this paper we are going to describe the process in another manner which is much simpler than the above one.

Let S be a subclass of the class C of all ordinal numbers which is confinal to C. If the elements of S are arranged by magnitude then we say that $S=\{\sigma_\xi\}_{\xi\in C}$ is a *confinal sequence.* An element σ_ξ of S is called a *fixed point* of S if $\sigma_\xi=\xi$.

First we define by transfinite induction the concept of figures. We define a figure of order 0 as a confinal sequence. Let now $\beta>0$ be a given ordinal number and suppose that the figures of order smaller than β have been already defined. We define the figures of order β as follows:

1) in the case of $\beta=\eta+1$ we define a figure of order β as a sequence of type C of distinct figures of order η;

2) in the case of a limit number β we define a figure of order β as a sequence of type β the ξth element of which is a figure of order ξ.

If $F=\{F_\tau\}$ is a figure then we say that F and the elements F_τ of F are components of F, the components of any component of F are components of F, as well. If a component G of F is a figure of order τ then we say that G is a component of order τ.

We associate with every figure F an element $a(F)$ and two figures $A(F)$ and $A(a(F))$ of order 0, furthermore with every sequence $S_\beta=\{F_\eta\}_{\eta\in\beta}$ of figures F_η of order ξ or with every sequence $S_\beta=\{F_\eta\}_{\eta\in\beta}$ of figures of order η a figure $A(S_\beta)$ of order 0. Now we define $a(F)$, $A(F)$, $A(a(F))$ and $A(S_\beta)$ as follows.

a) Let $a(F)$ be the smallest of the elements of C which belongs to F.

b) Let $F=\{F_\xi\}_{\xi\in\varkappa}$ be a figure of order β (where $\varkappa=C$ or $\varkappa$ is a limit number (and in this case $\varkappa=\beta$)). Let $A(a(F))$ be one of the components of order 0 of F the smallest element of which is $a(F)$. We define $A(F)$ as follows:

b$_1$) if $\beta=0$ then let $A(F)=F$

b$_2$) if $\beta=\eta+1$ then let $A(F)$ be the sequence S of the distinct $a(F_\xi)$ $(\xi\in C)$ arranged according to their magnitude, provided that S is a confinal sequence; otherwise let $A(F)$ be an arbitrary figure of order 0;

b$_3$) if β is a limit number then let $A(F)$ be the sequence of all the distinct elements, arranged in their magnitude, which belong to the intersection of all $A(F_\xi)$ with $\xi<\beta$, provided that this is a confinal sequence; otherwise let $A(F)$ be an arbitrary figure of order 0;

c) let $S_\beta=\{F_\xi\}_{\xi\in\beta}$ be a given sequence of type β of figures F_ξ of order τ or a given sequence of type β of figures F_ξ of order ξ. We define $A(S_\beta)$ as follows:

c$_1$) if $\beta=\eta+1$ then let $A(S_\beta)=A(F_\eta)$;

c$_2$) if β is a limit number then let $A(S_\beta)$ be the sequence of all the distinct elements, arranged in their magnitude, which belong to the intersection of all $A(F_\xi)$ with $\xi<\beta$, provided that this is a confinal sequence; otherwise let $A(S_\beta)$ be an arbitrary figure of order 0.

If a figure of order 0 is a stationary subclass of C then we call it a stationary figure of order 0; otherwise we call it a non-stationary figure of order 0. Similarly, if the components of order 0 of a figure F of order β are stationary then we call it a stationary figure of order β. Let $S=\{\sigma_\xi\}_{\xi\in C}$ be a stationary figure of order 0. If we associate with every σ_ξ its index ξ we obtain a strictly divergent function g on S for which $g(\gamma)\leqq\gamma$ holds. Thus it follows from Theorem 1 that the class $\{\sigma_\xi\in S: \sigma_\xi>\xi\}$ is non-stationary.

If S is a stationary figure of order 0 then we denote by $(S)'$ the figure of order 0 consisting of all the fixed elements of S. Clearly, $S-(S)'$ is non-stationary.

Let $\gamma>0$ be an arbitrary ordinal number, $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ a sequence of figures of order 0, and G a figure of order 0. We say that S_γ is coincident with G if, in the case of $\gamma=\eta+1$, $(A(S_\gamma))'=G$ and, in the case of a limit number γ, $A(S_\gamma)=G$. Let now $\beta>0$ be a given ordinal number, τ an arbitrary ordinal number, $0<\tau<\beta$, and suppose that the coincidence of a sequence $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ of figures F_ξ of order τ, where γ is an arbitrary ordinal number (and a sequence $S_\tau=\{F_\xi\}_{\xi\in\tau}$ of figures F_ξ of order ξ) with a figure G of order τ has been already defined. Let $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ be a sequence of figures F_ξ of order β, where γ is an arbitrary ordinal number, and $R_\beta=\{F_\xi\}_{\xi\in\beta}$ a sequence of figures F_ξ of order ξ, furthermore G a figure of order β. We say that S_γ is coincident with G if, in the case of $\gamma=\eta+1$, $(A(S_\gamma))'=$ $=A(a(G))$ and, if, in the case of a limit number γ, $A(S_\gamma)=A(a(G))$. Similarly, R_β is coincident with G if, in the case of $\beta=\eta+1$, $(A(R_\beta))'=A(a(G))$ and if, in the case of a limit number β, $A(R_\beta)=A(a(G))$. If F is a figure of order 0 or if $F=\{F_\xi\}_{\xi\in\varkappa}$ is a figure of order $\beta>0$ (where $\varkappa=C$ or $\varkappa$ is a limit number (and in this case $\varkappa=\beta$)) and, for every $\xi\in\varkappa$, the sequence $S_\xi=\{F_\zeta\}_{\zeta\in\xi}$ is coincident with F_ξ, then we say that F is connected. If F is a figure and all its components are connected, we say that F is perfect. It is clear that $A(a(F))$ is uniquely determined for any perfect figure F.

Now, we define by transfinite induction the operations $\mathbf{\Gamma}_\beta$ for $\beta\in C$. The operation $\mathbf{\Gamma}_0$ chooses a stationary figure $F^{(0)}$ of order 0. Let now $\beta>0$ be a given ordinal number and suppose that the figures $F^{(\xi)}$ of order ξ and the operation $\mathbf{\Gamma}_\xi$ have been already defined for every $\xi<\beta$. If $\beta=\eta+1$ then the operation $\mathbf{\Gamma}_\beta$ chooses a perfect stationary figure $F^{(\beta)}$ of order β such that $F^{(\eta)}$ is coincident with $F^{(\beta)}$. If β is a limit number then $\mathbf{\Gamma}_\beta$ chooses a perfect stationary figure $F^{(\beta)}$ of order β such that the figure $G^{(\beta)}=\{F^{(\xi)}\}_{\xi\in\beta}$ is coincident with $F^{(\beta)}$.

We shall prove that we have defined the figures $F^{(\xi)}$ of order ξ and the operations Γ_ξ for every $\xi \in C$.

We define the operation Φ as follows. Let $F = F^{(\beta)}$. Then $\Phi(F)$ is the figure $D^{(\beta)}$ of order β obtained in the following way: We omit the fixed elements from every component of order 0 of F, furthermore if $0 \leqq \tau < \beta$ and G is any component of order $\tau+1$ of F then we omit the fixed elements of $A(G)$.

Now we prove the following

Theorem A. *The class of all the elements of C belonging to $D^{(\beta)} = \Phi(F^{(\beta)})$ is non-stationary; β is an arbitrary element of C.*

Proof. Let $H = \{h_\xi\}_{\xi \in C}$ any component of order 0 of $D^{(\beta)}$. By definition $\xi < h_\xi$ for every $\xi \in C$. We define a function g on H by writing $g(h_\xi) = \xi$. Since g is strictly divergent and regressive, Theorem I implies that H is non-stationary. Thus, every component of order 0 of $D^{(\beta)}$ is non-stationary. Let now $0 < \gamma \leqq \beta$ and suppose that the class of all elements belonging to any component $G^{(\varrho)}$ of order $\varrho < \gamma$ of $D^{(\beta)}$ is non-stationary. Let $G^{(\gamma)} = \{F_\xi\}_{\xi \in \tau}$ where $\tau = C$ or τ is a limit number (and in this case $\tau = \gamma$). We denote by U_ξ the class of all elements of C which belong to F_ξ. Clearly the classes U_ξ $(\xi \in C)$ are mutually disjoint. Let $\gamma = \eta + 1$ and $A(G^{(\gamma)}) = \{\sigma_\xi\}_{\xi \in C}$. We split $A(G^{(\gamma)})$ into the union of two disjoint classes:

$$A(G^{(\gamma)}) = A_1 \cup A_2,$$

where $A_1 = \{\sigma_\xi : \xi \in C,\ \xi < \sigma_\xi\}$ and $A_2 = \{\sigma_\xi : \xi \in C,\ \xi = \sigma_\xi\}$. One can easily see that the smallest element of U_ξ in the case of $\sigma_\xi = \xi$ is greater than $a(F_\xi)$. Thus the class of the smallest elements of U_ξ, where $\xi = \sigma_\xi$, is non-stationary. On the other hand, the class of the smallest elements of U_ξ, where $\xi < \sigma_\xi$, is non-stationary, too. Since the classes U_ξ are mutually disjoint, Theorem II implies that the classes

$$U^{(1)} = \bigcup_{\substack{\xi \in C \\ \xi < \sigma_\xi}} U_\xi \quad \text{and} \quad U^{(2)} = \bigcup_{\substack{\xi \in C \\ \xi < \sigma_\xi}} U_\xi$$

are non-stationary. Consequently, by Theorem III, the class $U = U^{(1)} \cup U^{(2)}$ is non-stationary. If γ is a limit number then $B = \{a(F_\xi)\}_{\xi \in \tau}$ is non-stationary, because B is not confinal to C. Thus by the hypothesis and Theorem III, the class of elements of C belonging to $G^{(\gamma)}$ is non-stationary in the case of the limit number γ, as well. The theorem is proved.

Theorem B. *If $\beta \in C$ then there is a non-stationary class T_β such that $A(F^{(0)}) = A(F^{(\beta)}) \cup T_\beta$.*

Proof. We use transfinite induction. The theorem is obviously true for $\beta = 0$. Let $\beta > 0$ and suppose that the theorem is true for every $\gamma < \beta$. Put $F^{(\beta)} = \{F_\tau\}$ and $D^{(\beta)} = \Phi(F^{(\beta)})$. Let us denote by $U^{(\beta)}$ the class of all elements of C which belong to $D^{(\beta)}$. First we consider the case of $\beta = \eta + 1$. In this case

$$A(F^{(\eta)}) = A(F^{(\beta)}) \cup U^{(\beta)} \cup [A(F^{(\eta)}) - (A(F^{(\eta)}))'].$$

By the hypothesis

$$A(F^{(0)}) = A(F^{(\eta)}) \cup U^{(\beta)} \cup [A(F^{(\eta)}) - (A(F^{(\eta)}))'] \cup T_\eta,$$

where T_η is non-stationary. Since $U^{(\beta)}$ and $A(F^{(\beta)})-(A(F^{(\beta)}))'$ are non-stationary, the theorem is true for $\beta=\eta+1$, as well. Let now β be a limit number. Since

$$A(F^{(0)})-A(F^{(\beta)}) = A(F^{(0)})-\bigcap_{\gamma<\beta} A(F^{(\gamma)}) = \bigcup_{\gamma<\alpha} T_\gamma,$$

the theorem follows from the hypothesis and Theorem III.

Theorem C. *The class* $A(F^{(0)})-\{a(F^{(\beta)}): \beta\in C\}$ *is non-stationary.*

We omit the proof.

References

[1] G. Fodor, On a process concerning inaccessible cardinals. I, *Acta Sci. Math.*, 27 (1966), 111—124.
[2] G. Fodor, On a process concerning inaccessible cardinals. II, *Acta Sci. Math.*, 27 (1966), 129—140.

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