Homomorphisms, higher derivations, and derivations on associative algebras

By JOHN BORIS MILLER in Clayton (Victoria, Australia)

1. Introduction

Let $\mathfrak{A}$ be an algebra over a field K of characteristic zero. By a *derivation* D on $\mathfrak{A}$ we shall mean a mapping on $\mathfrak{A}$ into $\mathfrak{A}$ which is linear, that is $D(\lambda a+\mu b)=\lambda D(a)+\mu D(b)$ for all $a, b \in A$, $\lambda, \mu \in K$, and which satisfies the law

(1. 1) $$D(ab)=D(a)b+aD(b) \qquad (\text{all } a, b, \in \mathfrak{A}).$$

By an *endomorphism* E we shall mean a linear map on $\mathfrak{A}$ into $\mathfrak{A}$ satisfying

(1. 2) $$E(ab)=E(a)E(b) \qquad (\text{all } a, b \in \mathfrak{A}).$$

The derivations on $\mathfrak{A}$ form a Lie algebra under the product $[D_1, D_2]=D_1D_2-D_2D_1$. The endomorphisms form a semigroup under composition.

If $\mathfrak{A}$ is a Banach algebra, and D is a given derivation, bounded, then the operator H defined by

(1. 3) $$H=\exp(D)$$

is a bounded endomorphism, in fact an automorphism. Here $\exp(D)$ is defined by the exponential series, convergent in the uniform operator topology, and the proof uses the Leibniz formula

$$D^n(ab)=\sum_{m=0}^{n}\binom{n}{m}D^{n-m}(a)D^m(b)$$

in an obvious manner. The formula (1. 3) is useful in a number of contexts, for example in the proof of the Singer—Wermer theorem [7], in the theory of semigroups of bounded automorphisms, and also in purely algebraic settings, such as in aspects of the structure theory of fields, and of Lie algebras ([4], Ch. IV; [3]).

The converse question arises, whether an arbitrary automorphism on the Banach algebra has a logarithm which is a derivation. If the problem is restated in terms of continuous groups of bounded automorphisms, the matter is straightforward: if the semigroup $\{E_\lambda: \lambda>0\}$ is continuous and $E_\lambda \to I$ as $\lambda \to 0$ in the uniform operator topology, then its infinitesimal generator is a bounded derivation D, $E_\lambda=\exp(\lambda D)$ and the semigroup is embeddable in the group $\{E_\lambda: \lambda \text{ real}\}$. The proof is an immediate consequence of the definition of the generator. (The infinitesimal generator of a strongly continuous semigroup of endomorphisms is likewise a derivation.) Thus there is a natural one-to-one correspondence between uniformly continuous groups of automorphisms and bounded derivations (see G. HOCHSCHILD [2]).

The further question can be asked, whether a single arbitrary endomorphism can be represented in some fashion in terms of derivations and the exponential function, even if it cannot be embedded in a semigroup. This matter is not settled here, though it is hoped that the present investigation, which it prompted, may contribute to its solution. We consider instead certain homomorphisms on an arbitrary algebra $\mathfrak{A}$ into algebras of polynomials and power series with coefficients in $\mathfrak{A}$, and obtain formulae for these homomorphisms which are, formally at least, of the form (1. 3). These can alternatively be viewed as representation formulae for higher derivations in terms of derivations; they are not immediately concerned with normed algebras.

To be more specific, we need some notation and terminology. Given $\mathfrak{A}$ and $k \geqq 0$, let $\mathfrak{A}_k[t]$ denote the algebra of all polynomials $a_0+a_1t+a_2t^2+\ldots+a_kt^k$, of degree $\leqq k$ in an indeterminate t, with coefficients in $\mathfrak{A}$. Addition and scalar multiplication are defined in the obvious manner; the product of two polynomials in $\mathfrak{A}_k[t]$ is formed by multiplying the polynomials as usual, letting t commute with the elements of $\mathfrak{A}$, and then deleting all terms containing powers of t greater than k. That is $\mathfrak{A}_k[t] \cong \mathfrak{A}[t]/(t^{k+1})$, the residue class algebra of the algebra of all polynomials modulo the principal ideal (t^{k+1}). We consider those homomorphisms H from $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ which have the special property ε that a_0 in the image of a equals a, i. e.

$$H(a) = a+a_1t+a_2t^2+\ldots+a_kt^k. \tag{1.4}$$

(For brevity we shall sometimes call these ε-homomorphisms.) Given such H, the determination of $a_1, a_2, \ldots, a_k$ from a is linear; thus H determines k linear mappings $F_1, F_2, \ldots, F_k$ by $a_n=F_n(a)$, $n=1, 2, \ldots, k$, so that

$$H(a) = a+tF_1(a)+t^2F_2(a)+\ldots+t^kF_k(a). \tag{1.5}$$

Substituting this form in (1. 2) gives the set of identities

$$F_1(ab) = F_1(a)b+aF_1(b), \tag{1.6}$$

$$F_2(ab) = F_2(a)b+F_1(a)F_1(b)+aF_2(b), \tag{1.7}$$

$$F_3(ab) = F_3(a)b+F_2(a)F_1(b)+F_1(a)F_2(b)+aF_3(b), \tag{1.8}$$

and in general, for $n=1, 2, \ldots, k$,

$$F_n(ab) = \sum_{r=0}^{n} F_{n-r}(a)F_r(b) \qquad (F_0=I). \tag{1.9}$$

A sequence of operators $(I, F_1, F_2, \ldots, F_k)$ satisfying these equations for all $a, b \in \mathfrak{A}$ is called by JACOBSON ([4], p. 191) a *higher derivation of rank k*. Equation (1. 5) establishes a one-to-one correspondance between homomorphisms with the property ε and higher derivations. Similarly, considering homomorphisms from $\mathfrak{A}$ into the algebra $\mathfrak{A}_\infty[t]$ of all formal power series in t, we obtain *higher derivations* $\{I, F_1, F_2, \ldots\}$ *of infinite rank*, the F's satisfying (1. 9) for $n=1, 2, \ldots$.

The result to be proved, for both finite and infinite cases, states that, under a suitable matrix representation, H and so the corresponding higher derivation can be represented uniquely by a matrix $\exp(\boldsymbol{D})$, where the matrix $\boldsymbol{D}$ has derivations for its elements. In §§ 5, 6 we mention some applications to the case when $\mathfrak{A}$ is a Banach algebra, and also obtain a result on homomorphisms on commutative Banach algebras by related arguments.

2. Higher derivations for $A_k[t]$

Equations (1. 9) can be used to express the higher derivation $\{I, F_1, F_2, \ldots, F_k\}$ in terms of derivations. Clearly from (1. 6), F_1 is a derivation; write $F_1 = D_1$. Consider (1. 7), with $F_1 = D_1$. One solution for F_2 is $F_2 = \frac{1}{2} D_1^2$; moreover any two solutions for F_2 differ by a derivation, and any solution plus a derivation is again a solution. We may therefore take as the most general solution of (1. 6) and (1. 7)

$$F_1 = D_1, \quad F_2 = \frac{1}{2!} D_1^2 + \Omega_2 \tag{2. 1}$$

where D_1 and Ω_2 are arbitrary derivations. Turning next to equation (1. 8), supposing F_1 and F_2 given by (2. 1), we get as the most general solution, by similar arguments

$$F_3 = \frac{1}{3!} D_1^3 + \Omega_2 D_1 + \Omega_3 \tag{2. 2}$$

where Ω_3 is an arbitrary derivation. Similarly

$$F_4 = \frac{1}{4!} D_1^4 + \frac{1}{2} \Omega_2 D_1^2 + \Omega_3 D_1 + \frac{1}{2} \Omega_2^2 + \Omega_4, \tag{2. 3}$$

and so on. Let us call this 'the process P'. The formulae which arise in this way become increasingly complicated. We can clarify the nature of the process and obtain a general formula for F_n by means of the following matrix representation.

Let $\mathfrak{T}_{k+1}(\mathfrak{A})$ be the algebra of all $(k+1)\times(k+1)$ upper-triangular matrices with constant diagonals and elements from $\mathfrak{A}$, i. e. of the form

$$A_{k+1} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ & a_0 & a_1 & & \\ & & a_0 & & \\ & & & \ddots & \\ & & & & a_0 \end{pmatrix} \tag{2. 4}$$

where $a_j \in \mathfrak{A}$ for $j = 0, 1, \ldots, k$. Let $\mathfrak{D}_{k+1}(\mathfrak{A})$ be the subalgebra of $\mathfrak{T}_{k+1}(\mathfrak{A})$ consisting of all diagonal matrices; clearly $\mathfrak{A}$ is isomorphic to $\mathfrak{D}_{k+1}(\mathfrak{A})$, and

$$a \to A_{k+1} = \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix} \tag{2. 5}$$

is a faithful representation of $\mathfrak{A}$ in $\mathfrak{T}_{k+1}(\mathfrak{A})$. By mapping t to

$$t = \begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \tag{2. 6}$$

we extend this representation to an isomorphism between $\mathfrak{A}_k[t]$ and $\mathfrak{T}_{k+1}(\mathfrak{A})$, by which $a_0 + a_1 t + \ldots + a_k t^k$ is mapped to $\boldsymbol{A}_{k+1}$ of (2. 4).

Let $\mathfrak{B} \equiv \mathfrak{B}(\mathfrak{A})$ be the algebra of all linear operators on $\mathfrak{A}$. We map $\mathfrak{B}$ to $\mathfrak{D}_{k+1}(\mathfrak{B})$ as in (2. 5); and letting the matrices of $\mathfrak{T}_{k+1}(\mathfrak{B})$ operate on those of $\mathfrak{T}_{k+1}(\mathfrak{A})$ in the obvious fashion (that is, by formal application of matrix multiplication), we obtain a representation

$$H \to \boldsymbol{F}_{k+1} = \begin{pmatrix} I & F_1 & F_2 & \cdots & F_k \\ & I & F_1 & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{pmatrix} \tag{2. 7}$$

of the ε-homomorphism $H: \mathfrak{A} \to \mathfrak{A}_k[t]$ in $\mathfrak{T}_{k+1}(\mathfrak{B})$. Clearly $\boldsymbol{F}_{k+1}$ is a homomorphism on $\mathfrak{D}_{k+1}(\mathfrak{A})$ into $\mathfrak{T}_{k+1}(\mathfrak{A})$. One can verify further that these elements of $\mathfrak{T}_{k+1}(\mathfrak{B})$ give the matrix representations of all the endomorphisms H on $\mathfrak{A}_k[k]$ of the form

$$H(a + a_1 t + a_2 t^2 + \ldots + a_k t^k) = a + b_1 t + b_2 t^2 + \ldots + b_k t^k.$$

In fact, the matrices of linear operators mapping $\mathfrak{T}_{k+1}(\mathfrak{A})$ into $\mathfrak{T}_{k+1}(\mathfrak{A})$ (i. e. preserving leading diagonals) are precisely the elements of $\mathfrak{T}_{k+1}(\mathfrak{B})$, and the ε-endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are precisely the matrices $\boldsymbol{F}_{k+1}$ with elements F_n satisfying (1. 9). (By an ε-endomorphism we mean an endomorphism which preserves leading diagonals.)

Letting $D_1, D_2, \ldots, D_k$ be derivations on $\mathfrak{A}$, write

$$\boldsymbol{D}_{k+1} = \begin{pmatrix} 0 & D_1 & D_2 & \cdots & D_k \\ & 0 & D_1 & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}. \tag{2.8}$$

Then our principal result in the finite case is the second half of:

Theorem (k finite). *The derivations $D_1, D_2, \ldots, D_k$ in $\mathfrak{B}(\mathfrak{A})$ being given,* $\exp(\boldsymbol{D}_{k+1})$ *is the matrix in $\mathfrak{T}_{k+1}(\mathfrak{B})$ of an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with $F_1 = D_1$; that is,* $\exp(\boldsymbol{D}_{k+1})$ *is an ε-endomorphism on $\mathfrak{T}_{k+1}(\mathfrak{A})$.*

Conversely, given an ε-homomorphism H on $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, its matrix $\boldsymbol{F}_{k+1}$ can be written

$$\boldsymbol{F}_{k+1}=\exp(\boldsymbol{D}_{k+1}) \tag{2.9}$$

where $D_1=F_1, D_2, \ldots, D_k$ are derivations in $\mathfrak{B}(\mathfrak{A})$. This representation of H is unique.

Thus the ε-endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are precisely the matrices of the form $\exp(\boldsymbol{D}_{k+1})$.

Proof. Suppose that the derivations are given. It is not difficult to show that $\boldsymbol{D}_{k+1}$ is then a derivation on $\mathfrak{T}_{k+1}(\mathfrak{A})$. Moreover $\boldsymbol{D}_{k+1}$ is nilpotent, so that $\exp(\boldsymbol{D}_{k+1})$ is well defined in $\mathfrak{T}_{k+1}(\mathfrak{B})$. Moreover it is an ε-endomorphism on $\mathfrak{T}_{k+1}(\mathfrak{A})$: the proof of this fact is algebraically the same as for the Banach-algebra case mentioned in § 1, while questions of convergence do not arise. If $\exp(\boldsymbol{D}_{k+1})$ is written in the form (2. 7), it is easily verified that $F_1=D_1$. This proves the first part. (We also find

$$F_2=\frac{1}{2!}D_1^2+D_2,\quad F_3=\frac{1}{3!}D_1^3+\frac{1}{2}(D_1D_2+D_2D_1)+D_3,\ \ldots. \tag{2.10}$$

Thus the process P generates the derivations $D_2, D_3, \ldots$ explicitly only if the added arbitrary derivations Ω are given suitable forms, that is, if we write

$$\Omega_2=D_2,\quad \Omega_3=\frac{1}{2}(D_1D_2-D_2D_1)+D_3,\ \ldots; \tag{2.11}$$

note that the first term in Ω_3 here is a derivation.)

We prove the second part of the theorem by induction. The truth of the assertion for small values of k can be verified using (2. 10). Let l be an integer $\geqq 2$, and assume the converse statement in the theorem for $k=l-1$. Let $\boldsymbol{F}_{l+1}$ be the matrix of an ε-homomorphism, its elements $I, F_1, \ldots, F_l$ forming the corresponding higher derivation. Then the submatrix $\boldsymbol{F}_l$ is a ε-endomorphism on $\mathfrak{T}_l(\mathfrak{A})$, and so by hypothesis there exist derivations $D_1=F_1, D_2, \ldots, D_{l-1}$ making up a matrix $\boldsymbol{D}_l$ such that $\boldsymbol{F}_l=\exp(\boldsymbol{D}_l)$. Partition $\boldsymbol{F}_{l+1}$ into blocks as

$$\boldsymbol{F}_{l+1}=\begin{pmatrix}\boldsymbol{F}_l & f\\ o' & I\end{pmatrix}$$

where f is $l\times 1$ and o' is $1\times l$. Let Δ be an arbitrary derivation, and write

$$\boldsymbol{\Delta}_{l+1}=\begin{pmatrix}0 & D_1 & D_2 & \cdots & D_{l-1} & \Delta\\ & 0 & D_1 & & D_{l-2} & D_{l-1}\\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & D_1\\ & & & & & 0\end{pmatrix}=\begin{pmatrix}\boldsymbol{D}_l & \omega\\ o' & 0\end{pmatrix}.$$

Then $\boldsymbol{\Delta}_{l+1}$ is a derivation in $\mathfrak{T}_{l+1}(\mathfrak{B})$, and therefore

$$\exp(-\boldsymbol{\Delta}_{l+1}), \ =\begin{pmatrix}\exp(-\boldsymbol{D}_l) & g\\ o' & I\end{pmatrix},$$

is an ε-endomorphism in $\mathfrak{T}_{l+1}(\mathfrak{B})$. (The forms of f, ω and g are not important to the argument.) Consider the product $\exp(-\mathbf{\Delta}_{l+1})\boldsymbol{F}_{l+1}$; its blocked form is

$$\begin{pmatrix} \boldsymbol{I}_l & h \\ o' & I \end{pmatrix},$$

and since it belongs to $\mathfrak{T}_{l+1}(\mathfrak{B})$ and has constant diagonals, its elements are those of $\boldsymbol{I}_{l+1}$ except possibly for its $(1, l+1)$ element, which is an unknown operator, Ψ say. But $\exp(-\mathbf{\Delta}_{l+1})\boldsymbol{F}_{l+1}$ is a homomorphism on $\mathfrak{D}_{l+1}(\mathfrak{A})$ into $\mathfrak{T}_{l+1}(\mathfrak{A})$: by writing out the corresponding form of (1. 2) we find that Ψ is a derivation. Let $\mathbf{\Psi}$ be the $(l+1)\times(l+1)$ matrix having Ψ in the $(1, l+1)$ position and all other elements 0. Then we have

$$\boldsymbol{F}_{l+1} = \exp(\mathbf{\Delta}_{l+1})(\boldsymbol{I}_{l+1} + \mathbf{\Psi}_{l+1}) = \exp(\mathbf{\Delta}_{l+1})\exp(\mathbf{\Psi}_{l+1}) = \exp(\mathbf{\Delta}_{l+1} + \mathbf{\Psi}_{l-1})$$

since $\mathbf{\Delta}_{l+1}$ and $\mathbf{\Psi}_{l+1}$ clearly commute. Thus we have

$$\boldsymbol{F}_{l+1} = \exp(\boldsymbol{D}_{l+1})$$

with $D_l = \Delta + \Psi$, a derivation. The result follows by induction.

It remains to prove uniqueness. Given H and so $F_1, F_2, \ldots, F_k$, we see that D_1 is determined, and D_2 also by (2. 10). Suppose that for some j, $2 < j \leqq k$, $D_1, D_2, \ldots, D_{j-1}$ are determined. Now

$$F_j = (1, j+1) \quad \text{element of} \quad \boldsymbol{I} + \boldsymbol{D}_{k+1} + \frac{1}{2!}\boldsymbol{D}_{k+1}^2 + \ldots;$$

the term $+D_j$ appears once on the right hand side, as the contribution of $\boldsymbol{D}_{k+1}$, and the other powers of $\boldsymbol{D}_{k+1}$ contribute expressions containing only D_{j-1}, $D_{j-2}, \ldots, D_1$. Therefore D_j is determined. Uniqueness follows.

Corollary. *The endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are all automorphisms; they form a group under composition.*

We remark that the process P gives a certain prominence among the derivations to D_1. It is possible in fact to prove an alternative formula $\boldsymbol{F}_{k+1} = \exp(\mathbf{\Omega}_{k+1})\boldsymbol{G}_{k+1}$, with

$$\boldsymbol{G}_{k+1} = \begin{pmatrix} I & D_1 & D_1^2/2! \ldots D_1^k/k! \\ & I & D_1 \\ & & I \\ & & & \ddots \\ & & & & I \end{pmatrix} = \exp\begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ & 0 & D_1 \\ & & 0 \\ & & & \ddots \\ & & & & 0 \end{pmatrix}$$

and

$$\mathbf{\Omega}_{k+1} = \begin{pmatrix} 0 & 0 & \Omega_2 & \Omega_3 & \cdots & \Omega_k \\ & 0 & 0 & \Omega_2 \\ & & 0 & 0 \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

The proof is similar to that of the theorem. In geneial we have $\exp(\boldsymbol{\Omega}_{k+1})\boldsymbol{G}_{k+1} \neq \boldsymbol{G}_{k+1}\cdot\exp(\boldsymbol{\Omega}_{k+1})$, and the Ω's, though derivations, are again not those most obviously generated by the process P.

3. Higher derivations for $\mathfrak{A}_\infty[t]$

From the previous theorem we deduce the corresponding result for the infinite case, which we shall designate $k=\infty$. Here $\mathfrak{A}$ is represented by the diagonal matrices in $\mathfrak{T}_\infty(\mathfrak{A})$, the algebra of upper triangular matrices with constant diagonals and order ω. Taking the obvious definitions of sum, scalar multiple and product, we remark first that the product of two matrices in $\mathfrak{T}_\infty(\mathfrak{A})$ is always well defined: no element in the product involves more than finitely many non-zero elements from the factors. Thus $\mathfrak{T}_\infty(\mathfrak{A})$ is indeed an algebra. Again, if $\boldsymbol{D}$ has the form corresponding to (2. 8) for $k=\infty$, then although $\boldsymbol{D}$ may not be nilpotent, nevertheless $\exp(\boldsymbol{D})$ is well defined since each element of this matrix involves the sum of only finitely many elements in $\mathfrak{A}$.

We consider ε-endomorphisms on $\mathfrak{T}_\infty(\mathfrak{A})$; properties of ε-homomorphisms on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$ come by restriction to $\mathfrak{D}_\infty(\mathfrak{A})$.

Theorem ($k=\infty$). *The sequence $D_1, D_2, \ldots$ of derivations in $\mathfrak{B}(\mathfrak{A})$ being given,* $\exp(\boldsymbol{D})$ *is the matrix in $\mathfrak{T}_\infty(\mathfrak{B})$ of an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$, with $F_1=D_1$. Conversely, given an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$, its matrix $\boldsymbol{F}$ has the form*

$$\boldsymbol{F} = \exp(\boldsymbol{D}) \tag{3. 1}$$

where $D_1=F_1, D_2, \ldots$ are derivations. This representation is unique.

Thus the ε-endomorphisms on $\mathfrak{T}_\infty(\mathfrak{A})$ are precisely the matrices of the form $\exp(\boldsymbol{D})$.

Proof. For a matrix $\boldsymbol{X}$ in $\mathfrak{T}_k$ ($k\leqq\infty$) let $[\boldsymbol{X}]_j$ when $j\leqq k$ denote the element of the jth superdiagonal, counting the leading diagonal as first; and let $\boldsymbol{X}_j$ denote the leading $j\times j$ block in $\boldsymbol{X}$, so that $\boldsymbol{X}_j\in\mathfrak{T}_j$. Suppose $\boldsymbol{D}$ given, and $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{T}_\infty(\mathfrak{A})$. Then for positive integral j,

$$[\exp(\boldsymbol{D})(\boldsymbol{AB})]_j = \sum_{\alpha=1}^{j}(\exp(\boldsymbol{D}))_{1\alpha}(\boldsymbol{AB})_{\alpha j} =$$
$$= \sum_{\alpha=1}^{j}(\exp(\boldsymbol{D}_j))_{1\alpha}(\boldsymbol{A}_j\boldsymbol{B}_j)_{\alpha j} = [\exp(\boldsymbol{D}_j)(\boldsymbol{A}_j\boldsymbol{B}_j)]_j.$$

By the theorem for k finite, $\exp(\boldsymbol{D}_j)$ is an endomorphism. Thus $[\exp(\boldsymbol{D})(\boldsymbol{AB})]_j = [\exp(\boldsymbol{D})\boldsymbol{A}\cdot\exp(\boldsymbol{D})\boldsymbol{B}]_j$. So $\exp(\boldsymbol{D})$ is an endomorphism, clearly with the property ε.

Conversely, let $\boldsymbol{F}$ be an ε-endomorphism in $\mathfrak{T}_\infty(\mathfrak{B})$. Then

$$[\boldsymbol{F}(\boldsymbol{A})]_j = \sum_{\alpha=1}^{j}(\boldsymbol{F})_{1\alpha}(\boldsymbol{A})_{\alpha j} = \sum_{\alpha=1}^{j}(\boldsymbol{F}_j)_{1\alpha}(\boldsymbol{A}_j)_{\alpha j} = [\boldsymbol{F}_j(\boldsymbol{A}_j)]_j.$$

From this it follows that $\boldsymbol{F}_j$ is a ε-endomorphism in $\mathfrak{T}_j(\mathfrak{B})$ and so $\boldsymbol{F}_j=\exp(\boldsymbol{D}_j)$.

It is easily seen that $(\boldsymbol{D}_j)_i = \boldsymbol{D}_i$ for $i<j$; thus there is determined a unique $\boldsymbol{D} \in \mathfrak{T}_\infty(\mathfrak{B})$ such that

$$[\boldsymbol{F}(A)]_j = [\exp(\boldsymbol{D}_j)A_j]_j = [\exp(\boldsymbol{D})A]_j$$

for $j=1, 2, \ldots$. That is, $\boldsymbol{F} = \exp(\boldsymbol{D})$.

4. Inner derivations

The inner derivations on non-commutative $\mathfrak{A}$ are the operators D_c of the form

$$D_c(a) = ca - ac \qquad (a, c \in \mathfrak{A}).$$

They form an ideal in the Lie algebra of derivations. Some algebras are known to admit no derivations except inner deiivations. (For a recent summary of such results see [6]. To this can be added that S. Sakai and R. Kadison have separately found proofs that every von Neumann algebra admits no derivations except inner derivations. I must thank J. R. Ringrose for this information.)

If $D_1, D_2, \ldots$ in the theorem of §§ 2, 3 (take $k=\infty$ for definiteness) are all inner derivations, say

$$D_n(a) = c_n a - a c_n \qquad (\text{all } a \in \mathfrak{A};\ n = 1, 2, \ldots),$$

then it is easily seen that $\boldsymbol{D}$ is an inner derivation in $\mathfrak{T}_\infty(\mathfrak{B})$. In fact, write $\boldsymbol{C}$ for the matrix in $\mathfrak{T}_\infty(\mathfrak{A})$ having 0 in the leading diagonal and c_{n-1} in the nth diagonal, for $n=2, 3, \ldots$; then we find

$$\boldsymbol{D}(A) = \boldsymbol{C}A - A\boldsymbol{C} \qquad (\text{all} \quad A \in \mathfrak{T}_\infty(\mathfrak{A})).$$

and the Campbell—Hausdorff formula for $\mathfrak{T}_\infty(\mathfrak{A})$ gives

$$\exp(\boldsymbol{D})(A) = \exp(\boldsymbol{C})\, A \exp(-\boldsymbol{C}).$$

Thus in the case where $\mathfrak{A}$ admits only inner derivations we have the corollary that all ε-endomorphisms in $\mathfrak{T}_\infty(\mathfrak{B})$ are inner endomorphisms. (T is an inner endomorphism in $\mathfrak{B}(\mathfrak{A})$ if and only if for some regular $u \in \mathfrak{A}$, $Ta = uau^{-1}$ for all $a \in \mathfrak{A}$.)

5. Applications to Banach algebras

Let $\mathfrak{A}$ now be a complex Banach algebra, and $\mathfrak{B}(\mathfrak{A})$ the Banach algebra of all bounded linear operators on $\mathfrak{A}$ into $\mathfrak{A}$.

5. 1. For $\varrho>0$, write U_ϱ for the disc $\{\lambda : |\lambda| \leqq \varrho\}$ in the complex plane, and let $\mathfrak{F}_\varrho$ denote the algebra of all functions on U_ϱ into $\mathfrak{A}$ which are continuous on U_ϱ and holomorphic on the interior of U_ϱ. Each f in $\mathfrak{F}_\varrho$ is representable uniquely by its Taylor series about 0,

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots \qquad (a_n \in \mathfrak{A} \text{ for } n = 0, 1, \ldots).$$

$\mathfrak{F}_\varrho$ becomes a Banach algebra if it is normed by writing

$$\|f\|_\varrho = \max_{|\lambda| \leqq \varrho} \|f(\lambda)\| = \max_{|\lambda| = \varrho} \|f(\lambda)\|. \tag{5. 1}$$

An endomorphism H on $\mathfrak{F}_\varrho$ clearly has the algebraic properties of an endomorphism on $\mathfrak{A}_\infty[t]$, so that the theory of § 3 is relevant. If H is also a closed operator, then it is bounded. Suppose that this is the case, and that H has the property ε, so that (if we specify functions by their Taylor series) then

$$H(a_0+a_1\lambda+a_2\lambda^2+\dots) = b_0+b_1\lambda+b_2\lambda^2+\dots \tag{5. 2}$$

implies $b_0=a_0$. The property ε can alternatively be put in the form:

$$(Hf)(0)=f(0) \quad \text{for all} \quad f\in\mathfrak{F}_\varrho.$$

We consider the question of bounds. Suppose that $\{I, F_1, F_2, \dots\}$ is the higher derivation of infinite rank corresponding to H and making up $\boldsymbol{F}$, and $D_1, D_2, \dots$ is the sequence of derivations which occur in $\boldsymbol{D}$ in the formula (3. 1). The boundedness of H implies that of the F's and the D's. To prove this, apply H to the constant function $f(\lambda)\equiv a$, getting $H(a)=g$ where

$$g(\lambda) = a+c_1\lambda+c_2\lambda^2+\dots \qquad (|\lambda|\leqq\varrho),$$

say. Now the familiar Cauchy estimates for the coefficients of such a power series extend from the classical to the vector-function case ([1], p. 97), so we have, for $n=1, 2, \dots$

$$\|F_n(a)\| = \|c_n\| \leqq \frac{1}{\varrho^n}\max_{|\lambda|\leqq\varrho}\|g(\lambda)\| = \varrho^{-n}\|g\|_\varrho = \varrho^{-n}\|H(a)\|_\varrho \leqq \varrho^{-n}\|H\|\cdot\|a\|,$$

whence $\|F_n\|\leqq\varrho^{-n}\|H\|$. Thus the F's are bounded. Since $D_n=F_n+$a polynomial in $D_1, D_2, \dots, D_{n-1}$, it follows easily that the D's are bounded.

Note that non-trivial ε-endomorphisms on $\mathfrak{F}_\varrho$ exist if $\mathfrak{A}$ admits a non-zero derivation D: we have only to take $D_1=D$, $D_2=D_3=\dots=0$ in $\boldsymbol{D}$ and form $\exp(\boldsymbol{D})$.

5. 2. Consider the restriction of H to $\mathfrak{A}$ (which we also write H). Let $\varkappa$ be an arbitrary point of the disc U_ϱ, and $\sigma_\varkappa$ the evaluation map on $\mathfrak{F}_\varrho$ into $\mathfrak{A}$ at $\varkappa$; that is, $\sigma_\varkappa(f)=f(\varkappa)$ for $f\in\mathfrak{F}_\varrho$. Since $\sigma_\varkappa$ is a homomorphism, $\sigma_\varkappa H$ is an endomorphism on $\mathfrak{A}$, bounded if H is bounded; it clearly has a simple representation in terms of derivations, deducible from the formula (3. 1). It would therefore be interesting to have a characterization of the endomorphisms which can be factorized in the form $\sigma_\varkappa H$ for some $\varkappa$ and H, and to know for what $\mathfrak{A}$, if any, all endomorphisms in $\mathfrak{B}(\mathfrak{A})$ have this property.

5. 3. It is known that some algebras are sparingly supplied with derivations. Thus, if $\mathfrak{A}$ is a semisimple commutative Banach algebra, the Singer—Wermer theorem [7] shows that $\mathfrak{A}$ admits no bounded derivations other than 0. It follows from the theorem of § 3 and the discussion in § 5. 1 that for such $\mathfrak{A}$ the only bounded ε-homomorphisms on $\mathfrak{A}$ into $\mathfrak{F}_\varrho$, and on $\mathfrak{F}_\varrho$ into $\mathfrak{F}_\varrho$, are the injection mapping $H(a)=f$ with $f(\lambda)\equiv a$, and the identity mapping $H(f)=f$, respectively.

6. A related result

Some interesting and more general results can be obtained for Banach algebras by adapting the process P to the point derivations introduced by Singer and Wermer.

Let $\mathfrak{A}$ be a commutative Banach algebra over the complex field. Given a multiplicative linear functional φ on $\mathfrak{A}$, a *point derivation associated with* φ is a linear functional δ_φ satisfying

$$(6.5) \qquad \delta_\varphi(ab) = \delta_\varphi(a)\varphi(b)+\varphi(a)\delta_\varphi(b) \qquad (\text{all } a, b\in\mathfrak{A}).$$

We prove

Theorem. *If the complex commutative Banach algebra $\mathfrak{A}$ admits no non-zero point derivations, then the only homomorphisms on $\mathfrak{A}$ into $\mathfrak{H}_\varrho$, the algebra of all complex-valued functions continuous on the disc U_ϱ and holomorphic on its interior, are the mappings to constants, that is, the multiplicative linear functionals on $\mathfrak{A}$.*

(Note that the homomorphisms in the statement of the theorem are not required to have the property ε.)

Proof. Let H be a homomorphism on $\mathfrak{A}$ into $\mathfrak{H}_\varrho$. Then if $H(a)=f$, with

$$(6.2) \qquad f(\lambda) = a_0+a_1\lambda+a_2\lambda^2+\dots \qquad (|\lambda|\leqq\varrho;\ a_0, a_1, \dots \text{ complex}),$$

there is determined a sequence of linear functionals $\varphi_0, \varphi_1, \dots$ by

$$a_n=\varphi_n(a) \qquad (n=0, 1, \dots).$$

Substituting the form (6. 2) in (1. 2), we obtain a set of identities for the φ's like (1. 6)—(1. 9), namely

$$(6.3) \qquad \varphi_0(ab) = \varphi_0(a)\varphi_0(b),$$

$$(6.4) \qquad \varphi_1(ab) = \varphi_1(a)\varphi_0(b)+\varphi_0(a)\varphi_1(b),$$

$$(6.5) \qquad \varphi_2(ab) = \varphi_2(a)\varphi_0(b)+\varphi_1(a)\varphi_1(b)+\varphi_0(a)\varphi_2(b),$$

and in general for $n=1, 2, \dots$

$$\varphi_n(ab) = \sum_{\alpha=0}^{n} \varphi_{n-\alpha}(a)\varphi_\alpha(b).$$

Thus φ_0 is a multiplicative linear functional on $\mathfrak{A}$. Since φ_1 by (6. 4) is a point derivation on $\mathfrak{A}$ associated with φ_0, the assumption of the theorem implies that $\varphi_1=0$. But then φ_2 by (6. 5) is a point derivation, so $\varphi_2=0$. By induction we find that $\varphi_1=\varphi_2\dots=0$, so that $H(a)=\varphi_0(a)$.

Singer and Wermer give the following necessary and sufficient condition for the existence of point derivations in a commutative Banach algebra $\mathfrak{A}$ with identity. Given the multiplicative linear functional φ, write $M_\varphi=\{a\colon \varphi(a)=0\}$ for the corresponding maximal ideal in $\mathfrak{A}$, and M_φ^2 for the set of all linear combinations of squares of elements of M_φ, so that $M_\varphi^2\subseteqq M_\varphi$. Then non-zero point derivations associated with φ exist if and only if $M_\varphi^2\neq M_\varphi$.

Corollary. *The only homomorphisms from the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X into $\mathfrak{H}_\varrho$ are the multiplicative linear functionals.*

Proof. For every φ, $M_\varphi^2=M_\varphi$: for the details, see [7].

Note added in proof. R. J. Loy has extended the results of this paper under a weaker hypothesis than the ε property. See the following paper [8].

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Monash University, Melbourne

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