On homomorphisms of partially ordered semigroups

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In attempting to describe the order preserving homomorphisms of a partially ordered semigroup G onto a partially ordered semigroup G', it has proved necessary to impose conditions on G, on G', on the congruence ϱ determined by the homomorphism, or on a combination of these [7], [3], [1]. The approach used here is to assume that G/ϱ is residuated in such a way that each element of G/ϱ is both a left and a right residual of itself, and that the ϱ-class of each element of G contains a maximum element. Without assuming that G is residuated, as in [4], or even generalized residuated, [3], it is shown that if t is maximum in its ϱ-class, the residuals $t\,.\cdot a$ and $t\cdot .\,a$ exist for any $a \in G$, and that ϱ is determined by a subset of all such residuals.

When G/ϱ is a group the form of ϱ has been determined by Mme. DUBREIL-JACOTIN [1]; since a group is residuated, her result may be deduced from those described here. As an extension of this, the condition that G/ϱ be a group is replaced by the condition that G/ϱ be an integrally closed semigroup, and the structure of ϱ is then determined.

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Let G be a partially ordered set. That is, a set in which is defined a relation $\leqq$, which is reflexive, antisymmetric and transitive. For $x, y \in G$, the *greatest lower bound* of x and y, if it exists, is denoted by $x \wedge y$, and the *least upper bound*, if it exists, is denoted by $x \vee y$. An equivalence relation ϱ on G is called an *m-equivalence* if ϱ satisfies the following conditions:

(i) *for any $x \in G$, the ϱ-class $x\varrho$ of x contains a maximum element t_x,*

(ii) *for $x, y \in G$, $x \leqq y$ implies $t_x \leqq t_y$.*

The following notation will be used:

$$T(\varrho) = \{t \in G \mid t \text{ is maximum in its } \varrho\text{-class}\}.$$

For an equivalence relation ϱ satisfying (i), it is easily seen that (ii) is equivalent to:

(ii)′ *$x, y \in G$, $x < y$, $x \not\equiv y(\varrho)$, $x'\varrho x$ imply that there exists in G an element y' such that $y'\ \varrho\ y$ and $x' < y'$.* (Condition (ii)′ is the property (S) discussed in (2, 5).)

When ϱ is an m-equivalence, the set $G/\varrho = \{x\varrho \mid x \in G\}$ may be partially ordered by:

$$x\varrho \leqq y\varrho \quad \text{in} \quad G/\varrho \quad \text{if and only if} \quad t_x \leqq t_y \quad \text{in} \quad G.$$

We use the same notation for the partial orders in G and G/ϱ; it is clear that G/ϱ

is the order homomorphic image of G, in the sense that $x \leqq y$ in G implies $x\varrho \leqq y\varrho$ in G/ϱ. Note that $x\varrho \leqq y\varrho$ in G/ϱ if and only if there exist $x' \in x\varrho$ and $y' \in y\varrho$ such that $x' \leqq y'$. Further, the ϱ-classes are convex, for if $x, y, z \in G$ with $x \leqq y \leqq z$ and $x\varrho z$, then $t_x \leqq t_y \leqq t_z = t_x$ implies $t_x = t_y$, or $x\varrho y$.

A *partially ordered groupoid* is a partially ordered set G on which is defined a binary operation, which will be written multiplicatively, such that for $a, b, x \in G$, $a \leqq b$ implies $ax \leqq bx$ and $xa \leqq xb$. If multiplication is associative, G is called a *partially ordered semigroup*. If for $a, b \in G$ the set of all $x \in G$ such that $ax \leqq b$ ($xa \leqq b$) is non-empty and contains a maximum element, this element is called the *right (left) residual of b by a*, and is written $b \,.\!\cdot\, a$ ($b \cdot\!.\, a$). If $b \,.\!\cdot\, a$ ($b \cdot\!.\, a$) exists for all $a, b \in G$, then G is called *right (left) residuated*, and if G is both right and left residuated, it is said to be *residuated*.

A *congruence relation* on a partially ordered groupoid G is an equivalence relation ϱ on G which satisfies:

(iii) *for $x, y, z \in G$, $x \varrho y$ implies $xz \varrho yz$ and $zx \varrho zy$.*

An *m-congruence on G* is an m-equivalence on G which satisfies (iii).

When ϱ is an m-congruence, G/ϱ is a groupoid, and a homomorphic image of G, if multiplication in G/ϱ is defined by $a\varrho \cdot b\varrho = (ab)\varrho$. Further, G/ϱ is a partially ordered groupoid with the partial order defined above, for if $x\varrho, y\varrho, z\varrho \in G/\varrho$ with $x\varrho \leqq y\varrho$, then $t_x \leqq t_y$ implies $t_z t_x \leqq t_z t_y$, whence by (i) and (iii), $t_{zx} \leqq t_{zy}$; similarly for multiplication on the right.

Lemma 1. *Let ϱ be an m-congruence on a partially ordered groupoid G. Then G/ϱ is right residuated if and only if $t \,.\!\cdot\, a$ exists for every $t \in T(\varrho)$ and for every $a \in G$. In this case $t \,.\!\cdot\, a \in T(\varrho)$ $(t \,.\!\cdot\, a)\varrho = t\varrho \,.\!\cdot\, a\varrho$, and $a' \varrho a$ implies $t \,.\!\cdot\, a = t \,.\!\cdot\, a'$.*

Proof. *Sufficiency:* Let $a\varrho, b\varrho \in G/\varrho$, and let $a \in a\varrho$. Since $a(t_b \,.\!\cdot\, a) \leqq t_b$, it follows that $a\varrho(t_b \,.\!\cdot\, a)\varrho \leqq b\varrho$; on the other hand, if $a\varrho\, x\varrho \leqq b\varrho$ then $at_x \leqq t_b$, $t_x \leqq t_b \,.\!\cdot\, a$, $x\varrho \leqq (t_b \,.\!\cdot\, a)\varrho$. Hence $b\varrho \,.\!\cdot\, a\varrho$ exists, equal to $(t_b \,.\!\cdot\, a)\varrho$.

Necessity: Let $a \in G$, $t \in T(\varrho)$. Consider $t\varrho \,.\!\cdot\, a\varrho$ in G/ϱ, and let u be the maximum element in the class $t\varrho \,.\!\cdot\, a\varrho$. Then $a\varrho(t\varrho \,.\!\cdot\, a\varrho) \leqq t\varrho$ implies $au \leqq t$; but if $ax \leqq t$ then $a\varrho\, x\varrho \leqq t\varrho$, $x\varrho \leqq t\varrho \,.\!\cdot\, a\varrho$, $x \leqq t_x \leqq u$. Hence $t \,.\!\cdot\, a$ exists, equal to u. Since $t \,.\!\cdot\, a$ is the maximum element in $t\varrho \,.\!\cdot\, a\varrho$, it follows that if $a' \varrho\, a$ then $t \,.\!\cdot\, a' = t \,.\!\cdot\, a = u$.

Lemma 1 will be used as stated, but it may be noted that the following holds: the residual $b\varrho \,.\!\cdot\, a\varrho$ exists in G/ϱ if and only if $t_b \,.\!\cdot\, a$ exists for some (and hence all) $a \in a\varrho$.

It follows from Lemma 1 that if G/ϱ is right residuated, then for any $x \in G$, $t \,.\!\cdot\, x$ exists for any right residual $t[= t_b \,.\!\cdot\, a]$ of any element of $T(\varrho)$; for $t \in T(\varrho)$.

Residuals obey the following rules, quoted here without proof (see [2]); it is not necessary to assume that the groupoid G concerned is residuated, but only that the residuals concerned exist.

1. $b \leqq a \,.\!\cdot\, (a \cdot\!.\, b)$, *with equality if and only if* $b = a \,.\!\cdot\, x$ *for some* $x \in G$.
2. If G *is a semigroup,* $a \,.\!\cdot\, bc = (a \,.\!\cdot\, b) \,.\!\cdot\, c$ *and* $a \cdot\!.\, bc = (a \cdot\!.\, c) \cdot\!.\, b$.
3. $a \leqq b$ *implies* $a \,.\!\cdot\, c \leqq b \,.\!\cdot\, c$ *and* $c \,.\!\cdot\, b \leqq c \,.\!\cdot\, a$.

The existence of an identity element e in G, together with Rule 1, implies that $a = a \cdot\!.\, (a \,.\!\cdot\, a) = a \,.\!\cdot\, (a \cdot\!.\, a)$ for any $a \in G$, since $a = a \cdot\!.\, e = a \,.\!\cdot\, e$. However, even if G does not have an identity, it may still be true that $a=a^{\cdot}.(a.^{\cdot}a)=a.^{\cdot}(a^{\cdot}.a)$ for any $a\in G$; this is the case if and only if every element of G is both a left and a right residual of itself, in the sense that for every $a\in G$ there exists $x\in G$ such that $a=a^{\cdot}.x(a=a.^{\cdot}x)$. We shall call *self-residuated* a groupoid having this property (5).

Theorem 1. *Let G be a partially ordered semigroup, and let T be a non-empty subset of G satisfying, the following conditions:*

(α) *For any $t\in T$ and for any $a, x\in G$ there exist:*

$$t.^{\cdot}a,\quad t^{\cdot}.a,\quad \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a),\quad \Big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\Big).^{\cdot}x,\qquad \Big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\Big)^{\cdot}.x.$$

(β) *For any $t\in T$ and for any $a\in G$, $t.^{\cdot}a\in T$.*

(γ) *Each $t\in T$, and each $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$, for $a\in G$, is both a left and a right residual of itself.*

Define the relation ϱ_T on G by:

$$a\,\varrho_T\, b \quad \textit{if and only if}\quad t.^{\cdot}a=t.^{\cdot}b \quad \textit{for every}\quad t\in T.$$

Then ϱ_T is an m-congruence on G, and G/ϱ_T is residuated and self-residuated.

Conversely, if ϱ is an m-congruence on G such that G/ϱ is residuated and self-residuated, then $T(\varrho)$ satisfies (α), (β) and (γ), and $\varrho=\varrho_{T(\varrho)}$.

Proof. Clearly ϱ_T is an equivalence relation. For $a, x\in G$ and $t\in T$, $t.^{\cdot}a\in T$ (by (β)), and this implies, by (α), that $(t.^{\cdot}a).^{\cdot}x$ exists; by (α) again, $t.^{\cdot}ax$ exists, and then $(t.^{\cdot}a).^{\cdot}x=t.^{\cdot}ax$, each being the maximum $z\in G$ such that $axz\leqq t$. (It is here that we use the fact that G is a semigroup). Hence if $b\in G$ and $a\,\varrho_T\,b$, then

$$t.^{\cdot}ax=(t.^{\cdot}a).^{\cdot}x=(t.^{\cdot}b).^{\cdot}x=t.^{\cdot}bx,$$

$$t.^{\cdot}xa=(t.^{\cdot}x).^{\cdot}a=t'.^{\cdot}a=t'.^{\cdot}b=(t.^{\cdot}x).^{\cdot}b=t.^{\cdot}xb,$$

where $t'=t.^{\cdot}x\in T$, by (β). Thus ϱ_T is a congruence relation. To see that ϱ_T is an m-congruence, we note that by Rule 1, $a\leqq t^{\cdot}.(t.^{\cdot}a)$; this implies that

$$a\leqq \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\leqq t^{\cdot}.(t.^{\cdot}a)\quad \text{for any}\quad t\in T,$$

and since (by Rule 1 again), $t.^{\cdot}(t^{\cdot}.(t.^{\cdot}a))=t.^{\cdot}a$ it follows from Rule 2 that $t.^{\cdot}a=t.^{\cdot}(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a))$. That is, $a\,\varrho_T\,(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a))$. Clearly $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$ is maximum in $a\varrho_T$. Using Rule 3 twice, we see that $a\leqq c$ in G implies that $t^{\cdot}.(t.^{\cdot}a)\leqq t^{\cdot}.(t.^{\cdot}c)$ for any $t\in T$, and hence that $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\leqq \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}c)$, so that ϱ_T is indeed an m-congruence.

By (α), ϱ_T satisfies the conditions of Lemma 1, and so G/ϱ_T is residuated. For $a\varrho_T\in G/\varrho_T$, write $t_a=\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$; then

$$a\varrho_T{}^{\cdot}.(a\varrho_T.^{\cdot}a\varrho_T)=t_a\varrho_T{}^{\cdot}.(t_a\varrho_T.^{\cdot}t_a\varrho_T)=$$

$$=(t_a{}^{\cdot}.(t_a.^{\cdot}t_a))\varrho_T=t_a\varrho_T\quad (\text{using } (\gamma))=a\varrho_T,$$

and similarly $a\varrho_T=a\varrho_T.^{\cdot}(a\varrho_T{}^{\cdot}.a\varrho_T)$, so that G/ϱ_T is self-residuated.

Conversely, let G and ϱ be as stated; write $T=T(\varrho)$. Lemma 1 implies the existence of $t.\cdot a$ and $t^{\cdot}.a$, and that $t.\cdot a\in T$; for the rest, it is enough to show that each $t\in T$ is both a left and a right residual of itself, and that $\bigwedge_{t\in T} t^{\cdot}.(t.\cdot a)=t_a$.

Let $a\varrho\in G/\varrho$; since G/ϱ is self-residuated, we have from Lemma 1 that

$$a\varrho=t_a\varrho=t_a\varrho^{\cdot}.(t_a\varrho.\cdot t_a\varrho)=t_a\varrho^{\cdot}.(t_a.\cdot t_a)\varrho=(t_a^{\cdot}.(t_a.\cdot t_a))\varrho.$$

Since $t_a^{\cdot}.(t_a.\cdot t_a)$ is maximum in its ϱ-class (Lemma 1 again), we deduce $t_a=t_a^{\cdot}.\,.(t_a.\cdot t_a)$; similarly $t_a=t_a.\cdot(t_a^{\cdot}.t_a)$. By Rule 2, $t_a\leqq t^{\cdot}.(t.\cdot t_a)$ for any $t\in T$; in particular, $t_a=t_a^{\cdot}.(t_a.\cdot t_a)$. Hence $t_a=\bigwedge_{t\in T} t^{\cdot}.(t.\cdot t_a)$, and since, by Lemma 1 again, $t.\cdot t_a=t.\cdot a$, it follows that $t_a=\bigwedge_{t\in T} t^{\cdot}.(t.\cdot a)$.

By the first part of the Theorem, we now have that ϱ_T is an m-congruence on G such that G/ϱ_T is residuated and self-residuated, and it only remains to show that $\varrho=\varrho_T$. By Lemma 1, ϱ is finer than ϱ_T. Let $a\equiv b(\varrho_T)$; then with t_a maximum in $a\varrho$, $t_a.\cdot t_a=t_a.\cdot a=t_a.\cdot b=t_a.\cdot t_b$, (using Lemma 1), implies $t_b\,(t_a.\cdot t_a)\leqq t_a$, which in turn implies $t_b\leqq t_a^{\cdot}.(t_a.\cdot t_a)=t_a$, using ($\gamma$) for the last equality. Similarly $t_a\leqq t_b$, whence $t_a=t_b$; that is, $a\,\varrho\, b$. We conclude that $\varrho=\varrho_T$, and the theorem is proved.

Corollary 1.1. *If ϱ and σ are m-congruences on a partially ordered semigroup G such that G/ϱ and G/σ are residuated and self-residuated, then ϱ is finer than σ if and only if $T(\sigma)\subseteqq T(\varrho)$.*

Proof. If ϱ is finer than σ, any element of G maximum in its σ-class must be maximum in its ϱ-class. Conversely, if $T(\sigma)\subseteqq T(\varrho)$, then $a\,\varrho\, b$ implies $t.\cdot a=t.\cdot b$ for any $t\in T(\varrho)$, and therefore for any $t\in T(\sigma)$; by Theorem 1, $a\,\sigma\, b$.

Definition. An element a of a partially ordered groupoid G is called *equiresidual* if whenever one of $a.\cdot x$, $a^{\cdot}.x$ exists for $x\in G$, so does the other, and $a.\cdot x=a^{\cdot}.x$. We shall denote their common value by $a{:}x$.

Corollary 1.2. *Let G, ϱ be as in Theorem 1. Then G/ϱ is commutative if and only if each $t\in T(\varrho)$ is equiresidual.*

Proof. Let $a\varrho, b\varrho\in G/\varrho$, and suppose that each $t\in T(\varrho)$ is equiresidual. Then $b\varrho.\cdot a\varrho=(t_b.\cdot a)\varrho=(t_b^{\cdot}.a)\varrho=b\varrho^{\cdot}.a\varrho$, by Lemma 1. Since G/ϱ is residuated and self-residuated, $a\varrho\, b\varrho.\cdot a\varrho\, b\varrho=a\varrho\, b\varrho^{\cdot}.a\varrho\, b\varrho=(a\varrho\, b\varrho^{\cdot}.b\varrho)^{\cdot}.a\varrho=$ $=(a\varrho\, b\varrho.\cdot b\varrho)^{\cdot}.a\varrho=a\varrho\, b\varrho.\cdot b\varrho\, a\varrho$, using Rule 2, and so $b\varrho\, a\varrho\,(a\varrho\, b\varrho.\cdot a\varrho\, b\varrho)\leqq$ $\leqq a\varrho\, b\varrho$. Hence $b\varrho\, a\varrho\leqq a\varrho\, b\varrho^{\cdot}.(a\varrho\, b\varrho.\cdot a\varrho\, b\varrho)=a\varrho\, b\varrho$. Similarly $a\varrho\, b\varrho\leqq b\varrho\, a\varrho$, whence equality.

Conversely, if G/ϱ is commutative, $(t_b.\cdot a)\varrho=b\varrho.\cdot a\varrho=b\varrho^{\cdot}.a\varrho=(t_b^{\cdot}.a)\varrho$. Since each of $t_b.\cdot a$, $t_b^{\cdot}.a$ is maximum in its ϱ-class, equality follows.

It follows from the proof of Corollary 1.2 that a residuated, self-residuated semigroup G is commutative if and only if every element of G is equiresidual.

Note 1. If each $t\in T$ is equiresidual, and if G is residuated, (α) and (γ) are enough to ensure that ϱ_T in Theorem 1 is an m-congruence. Condition (β) was used only to show that ϱ_T is regular on the left with respect to multiplication; but for $a, b, x\in G$ with $a\equiv b(\varrho_T)$, we now have

$$t.\cdot xa=t^{\cdot}.xa=(t^{\cdot}.a)^{\cdot}.x=(t^{\cdot}.b)^{\cdot}.x=t^{\cdot}.xb=t.\cdot xb.$$

For a discussion of the case where T consists of a single equiresidual element in a residuated semigroup with identity, and where G/ϱ_T is a group, see MAURY [6].

Note 2. Condition (γ) is not necessary if G has an identity element.

Thus in a commutative, residuated semigroup G with identity, any non-empty subset T of G defines an $\mathring{m}$-congruence ϱ_T as in Theorem 1, provided only that for any $a \in G$, $\bigwedge_{t \in T} t^{\cdot}.(t{:}a)$ exists. Then G/ϱ_T is a residuated semigroup with identity; the maximum element in the ϱ_T class of a is $\bigwedge_{t \in T} t{:}(t{:}a)$. In particular, if x is a fixed element of G, let $T=\{x\}$, and write $\varrho_x=\varrho_{\{x\}}$. Then

$$a \varrho_x b \quad \text{if and only if} \quad x{:}a = x{:}b,$$

and $T(\varrho_x)=\{\bigwedge_{x \in T} x : (x : a), a \in G\}=\{x : (x : a), a \in G\}$, so that ϱ_x is MOLINARO's congruence relation A_x (7) (see below).

Note 3. There is a difference between the two parts of Theorem 1. Given that ϱ is an $\mathring{m}$-congruence such that G/ϱ is residuated and self-residuated, it follows that $\varrho=\varrho_{T(\varrho)}$, and that for any $a \in G$, $\bigwedge_{t \in T(\varrho)} t^{\cdot}.(t.^{\cdot}a) \in T(\varrho)$. Yet given $T \subseteqq G$ satisfying (α), (β) and (γ), to establish that ϱ_T is an $\mathring{m}$-congruence and that G/ϱ_T is residuated and self-residuated, it is not necessary to assume that $t_a=\bigwedge_{t \in T} t^{\cdot}.(t.^{\cdot}a)$ is in T for every $a \in G$, but only that t_a, $t_a{}^{\cdot}.x$ and $t_a.^{\cdot}x$ exist, for any $x \in G$. Then the set of elements maximum in their ϱ_T-classes is $T(\varrho_T)=\{\bigwedge_{t \in T} t^{\cdot}.(t.^{\cdot}a), \text{ for } a \in G\}$, of which T is a subset, in general a proper subset. Even the fact that $t_a \in T$ does not force the equality of T and $T(\varrho_T)$; in the semigroup $G=\{e, a, b, c, z\}$, with $e>a>c>z$, $e>b>c>z$ and $xy=x \wedge y$ for all $x, y \in G$, let $T=\{e, a, b\}$. Then G is a residuated, commutative semigroup with identity e; T satisfies (α), (β) and (γ), so Theorem 1 holds. Yet $T(\varrho_T)=\{e, a, b, c\}$, which properly contains T. Hence in general the representation of ϱ described in Theorem 1 is not unique.

Note 4. Although T is closed under residuation, in the sense that (β) holds, in general T is not closed under multiplication. In the example above, $a, b \in T$ but $ab=c \notin T$.

Note 5. Given ϱ satisfying the conditions of Theorem 1, it follows by symmetry that $T(\varrho)=T$ satisfies:

(α)′ *For any $t \in T$ and for any $a, x \in G$ there exist:*

$$t^{\cdot}.a,\quad t.^{\cdot}a,\quad \bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a),\quad \Big(\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)\Big)^{\cdot}.x \quad \text{and} \quad \Big(\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)\Big).^{\cdot}x.$$

(β)′ *For any $t \in T$ and for any $a \in G$, $t^{\cdot}.a \in T$.*

(γ)′ *Each $t \in T$, and each $\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)$, for $a \in G$, is both a left and a right residual of itself.*

Although this argument applies to $T(\varrho)$, it does not apply to any T satisfying (α), (β) and (γ); for example, T may satisfy (α) without satisfying (α)′. If T satisfies (α), (β) and (γ) as well as (α)′, (β)′ and (γ)′, then

$$a\,\varrho\,b,\; t\,.\cdot a=t\,.\cdot b \quad \text{for all} \quad t\in T,\; t\cdot.\,a=t\cdot.\,b \quad \text{for all} \quad t\in T,$$

are all equivalent.

Note 6. In [7], I. MOLINARO considered equivalence relations on a residuated semigroup S. He showed that for $t\in S$, the relation A_t (${}_tA$) defined by $a\equiv b(A_t)$ $(a\equiv b({}_tA))$ if and only if $t\,.\cdot a=t\,.\cdot b$ $(t\cdot.\,a=t\cdot.\,b)$ is an m-equivalence, regular on the right (left) with respect to multiplication.

If a subset T of a partially ordered semigroup G satisfies (α), (β) and (γ), one may still define the relation A_t as above, since, by (α), the residuals concerned exist; obviously A_t is an equivalence relation. Further, as in the proof of Theorem 1, A_t is regular on the right with respect to multiplication. Finally, $t\,.\cdot x=t\,.\cdot\{t\cdot.\,(t\,.\cdot x)\}$ implies that A_t is an m-equivalence, the maximum element in the class of $x\in G$ being $t\cdot.\,(t\,.\cdot x)$. By (γ), the maximum element in the class containing t is t itself. Thus an m-congruence ϱ on G, which satisfies the conditions of Theorem 1, may be expressed as the intersection of the m-equivalences A_t (${}_tA$) for $t\in T$, each A_t (${}_tA$) being regular on the right (left) with respect to multiplication.

When t is equiresidual A_t is a congruence relation, and several papers (cf. [4], [6], [7]) have been written about the situation where A_t is defined on residuated gerbiers, where by definition a gerbier is a semigroup G in which every two elements x, y have a least upper bound $x\vee y$ satisfying $a(x\vee y)=ax\vee ay$, $(x\vee y)a=xa\vee ya$ for all $x, y\in G$. If in addition every pair $x, y\in G$ have a greatest lower bound $x\wedge y$, G is called a lattice semigroup.

Note 7. If x and y are equiresidual elements of a residuated lattice semigroup G, then $\varrho=A_x\cap A_y$ is an m-congruence on G, with

$$T(\varrho)=\{t_a=x:(x:a)\wedge y:(y:a), \text{ for } a\in G\}.$$

For ϱ is certainly a congruence on G, while

$$a\leqq x:(x:a)\wedge y:(y:a)\leqq x:(x:a)$$

implies $a\equiv t_a(A_x)$, by convexity, and similarly $a\equiv t_a(A_y)$, so $a\,\varrho\,t_a$. Clearly $a\leqq t_a$, t_a is maximum in its ϱ-class, and $a\leqq b$ implies $t_a\leqq t_b$.

Example. Let $S=\{x|x$ is a real number and $x\leqq -2\}\cup\{-1\}\cup\{0\}$, with the usual ordering. If $x\leqq -2$, define $xy=yx=-2$ for any $y\in S$, and for $x, y>-2$, define $xy=yx=\min\{x, y\}$. Then S is a partially ordered semigroup without identity element. Let Z denote the integers under addition, with the usual ordering; as an ordered group, Z is residuated, with $i:j=i-j$ for $i, j\in Z$. Let $G=S\times Z$, with co-ordinatewise multiplication and ordering. For $a, b\in S$ and $a<-2$, there is no $x\in S$ such that $bx\leqq a$, so S is not residuated. It is easy to see that the direct product of residuated semigroups is residuated if and only if each factor is residuated, so G is not residuated. Yet for $x\in S$, $0:x=0$, $-1:0=-1$, $-1:x=0$ if $x\leqq -1$, $-2:0=-2$, $-2:-1=-2$, $-2:x=0$ if $x\leqq -2$. It follows that

$$T=\{(n, i)|\, n=-2,\ -1 \text{ or } 0,\ i\in Z\}$$

is a subset of G satisfying the first two parts of (α), (and satisfying (β)). Consider $a=(x,j)\in G$, for $x\leqq -2$. For $t=(n,i)\in T$, $t:(t:a)=(n,j)$ and so $\bigwedge_{t\in T} t:(t:a)=$ $=(-2,j)\in T$. For $a=(-1,j)$ and $t=(0,i)$ or $t=(-2,i)$, $t:(t:a)=(0,j)$, while for $t=(-1,i)$, $t:(t:a)=(-1,j)$. Thus $\bigwedge_{t\in T} t:(t:a)=(-1,j)\in T$. Similarly, for $a=(0,j)$, $\bigwedge_{t\in T} t:(t:a)=(0,j)\in T$. Hence T satisfies (α). Finally, for $t=(n,i)\in T$, $t:(t:t)=(n,i):(0,0)=(n,i)=t$, so T satisfies (γ). By Theorem 1, G/ϱ_T is a residuated, self-residuated semigroup. The formulae above show that the ϱ_T-classes consist of the points $\{(0,i)\}$ and $\{(-1,i)\}$, $i\in Z$, and the lines $\{(x,i)|x\leqq -2, i\in Z\}$. If $U=\{-2,-1,0\}$, with the usual ordering and $xy=yx=\min\{x,y\}$ for $x,y\in U$, then G/ϱ_T is isomorphic to the residuated, self-residuated semigroup $U\times Z$.

The formulae above also show that $A_{(n,i)}=A_{(n,j)}$ for any $i,j\in Z$, so that $\varrho_T=A_{(0,0)}\cap A_{(-1,0)}\cap A_{(-2,0)}$; since $A_{(-1,0)}\leqq A_{(0,0)}$ and $A_{(-2,0)}\leqq A_{(0,0)}$, in fact $\varrho_T=A_{(-2,0)}\cap A_{(-1,0)}$, though ϱ_T is not a congruence of the A type. Thus the representation of an m-congruence is not in general unique.

The situation illustrated in this example is typical of that in general. One may show that if $\varrho=\varrho_T$ and $T'\subseteqq T$ satisfies

$$\bigwedge_{t'\in T'} t'^{\cdot}.(t'.^{\cdot}a)=\bigwedge_{t\in T}(t^{\cdot}.(t.^{\cdot}a) \quad \text{for any } a\in G,$$

then $\varrho=\varrho_{T'}$, using the fact that $\varrho_{T'}$ is an m-congruence such that $T(\varrho_{T'})=T(\varrho)$.

II

A residuated semigroup G, with identity e, for which $a.^{\cdot}a=a^{\cdot}.a=e$ for every non-zero $a\in G$, is called *integrally closed.* We now investigate under what conditions a partially ordered semigroup G has an integrally closed homomorphic image, under the hypothesis that each congruence class contains a maximum element.

Let ϱ be an m-congruence on a partially ordered semigroup G such that G/ϱ is integrally closed. Then G/ϱ has an identity element $f\varrho$; let f be the element maximum in this class. Since G/ϱ is then self-residuated, Theorem 1 and its dual hold, and we have

(a) *$f.^{\cdot}a$ and $f^{\cdot}.a$ exist for any $a\in G$.*

Further, f satisfies the following conditions:

(b) *f is equiresidual.*

(c) *f is a residual of itself.*

(d) *$(f:a).^{\cdot}(f:a)=f=(f:a)^{\cdot}.(f:a)$ for any $a\in G$. In particular, $f=f:f$.*

For (b), $(f.^{\cdot}a)\varrho=f\varrho.^{\cdot}a\varrho$ (by Lemma 1) $=(a\varrho^{\cdot}.a\varrho).^{\cdot}a\varrho$ (since G/ϱ is integrally closed) $=(a\varrho.^{\cdot}a\varrho)^{\cdot}.a\varrho=f\varrho^{\cdot}.a\varrho=(f^{\cdot}.a)\varrho$; by Lemma 1, $f^{\cdot}.a=f.^{\cdot}a$, each being maximum in its class. The third condition follows at once from Theorem 1. Finally, for any $x\varrho\in G/\varrho$, $x\varrho.^{\cdot}x\varrho=f\varrho=x\varrho^{\cdot}.x\varrho$ implies that $t_x.^{\cdot}t_x=f=t_x^{\cdot}.t_x$; then $f:a\in T(\varrho)$ implies (d). In particular, $f=(f:f).^{\cdot}(f:f)=(f:(f:f))^{\cdot}.f=f:f$.

We have now proved the first part of the following Theorem.

Theorem 2. *A necessary and sufficient condition that there exist an m-con-*

gruence ϱ on a partially ordered semigroup G such that G/ϱ is integrally closed, is that G contain an element f satisfying (a), (b), (c) *and* (d).

For the second part, we require the following Lemma.

Lemma 2. *Let G, ϱ, $T(\varrho)$ be as in Theorem 1. Then G/ϱ has an identity element if and only if there exists $f \in G$ such that $t \cdot{}^{\cdot} f = t = t^{\cdot} \cdot f$ for every $t \in T(\varrho)$.*

Proof. The necessity is obvious, so let $f \in G$ be such that $t \cdot{}^{\cdot} f = t = t^{\cdot} \cdot f$ for all $t \in T(\varrho)$. Then for $a\varrho \in G/\varrho$, $a\varrho \cdot{}^{\cdot} f\varrho = (t_a \cdot{}^{\cdot} f)\varrho = t_a\varrho = a\varrho$ implies $f\varrho\, a\varrho \leqq a\varrho$. On the other hand, $a\varrho \leqq (f\varrho\, a\varrho) \cdot{}^{\cdot} f\varrho = f\varrho\, a\varrho$, so that $a\varrho = f\varrho\, a\varrho$. Similarly $f\varrho$ is a right identity for G/ϱ.

Proof of sufficiency of Theorem 2. Let $f \in G$ satisfy (a), (b), (c) and (d), and consider $T = \{t = f : a \mid a \in G\}$; we show that T satisfies the conditions of Theorem 1. First, for any $x, a \in G$, $f : a$ and $f : ax$ exist, and so therefore does $(f : a) \cdot{}^{\cdot} x = t \cdot{}^{\cdot} x$; similarly $t^{\cdot} \cdot x$ exists. Both these residuals are elements of T, so (β) is satisfied. Next, for any $y \in G$,

$$(f : y)^{\cdot} \cdot ((f : y) \cdot{}^{\cdot} a) = (f : y)^{\cdot} \cdot (f : ya) =$$
$$= (f : y)^{\cdot} \cdot ((f : a)^{\cdot} \cdot y) = f : \{(f : a)^{\cdot} \cdot y\} y \geqq f : (f : a).$$

But $f = (f : f) \cdot{}^{\cdot} (f : f) = (f : (f : f))^{\cdot} \cdot f = f : f$, so $(f : f)^{\cdot} \cdot \{(f : f) \cdot{}^{\cdot} a\} = f : (f : a)$, whence $f : (f : a) = \bigwedge_{t \in T} t^{\cdot} \cdot (t \cdot{}^{\cdot} a) \in T$. Condition ($\alpha$) follows at once. For ($\gamma$), we use Rule 2 and the fact that $f = f : f$ is equiresidual to obtain

$$t = f : a = (f : f) : a = f : fa = (f : a)^{\cdot} \cdot f = f : af = (f : a) \cdot{}^{\cdot} f.$$

By Theorem 1, $\varrho = \varrho_T$ is an m-congruence on G such that G/ϱ is residuated and self-residuated. To show that G/ϱ is integrally closed, we prove that in fact G/ϱ is a group. Since $a\,\varrho\,b$ if and only if $t_a = t_b$, and since $t_a = f : (f : a)$ (see above), we may use Rule 1 to obtain; $a\,\varrho\,b$ if and only if $f : a = f : b$. Then for any $t \in T$, $t \cdot{}^{\cdot} f = (f : a) \cdot{}^{\cdot} f = (f : f)^{\cdot} \cdot a = f : a = t = t^{\cdot} \cdot f$, and Lemma 2 shows that $f\varrho$ is the identity element of G/ϱ. Finally, $a(f : a) \equiv f(\varrho)$ for any $a \in G$, since $f : a(f : a) = (f : a) \cdot{}^{\cdot} (f : a) = f : f$, so G/ϱ is a group, and is a fortiori integrally closed. The Theorem is proved.

Since we do not require that G is a residuated semigroup, Theorem 2 generalizes the result of MAURY [6]. One may deduce from Theorem 2 the result ([1], p. 107), of Mme. DUBREIL-JACOTIN, that any m-congruence ϱ on G resulting in a group image G/ϱ is necessarily defined by: $a\,\varrho\,b$ if and only if $\langle f : a \rangle = \langle f : b \rangle = \{x \in G \mid ax \leqq f\}$, where f is the maximum element in the identity class of G/ϱ. See also L. FUCHS [3].

Note 8. It is not necessary to assume that f is idempotent. It does follow from $f = f : f$ that $f^2 \leqq f$, but it may happen that $f^2 < f$. Nevertheless f is the maximum element satisfying $x^2 \leqq x$ in G, for if $x^2 \leqq x$ then

$$f : x \leqq f : x^2 = (f : x) \cdot{}^{\cdot} x \text{ implies } (f : x)x \leqq f : x,$$

so $x \leqq (f : x)^{\cdot} \cdot (f : x) = f$.

If G has an identity element e, then $e\,\varrho\,f$ and $e \leqq f$. We note also (cf. [1], Theorem 5), that f is the maximum element of the form $x.\cdot x$ or $x^{\cdot}.x$ in G, for $f=f:f$, while $f=(f:x).\cdot(f:x)=(f:(f:x))^{\cdot}.x\geqq x^{\cdot}.x$.

In the example above, the element $f=(0, 0)$ satisfies (a), (b), (c) and (d), and G/ϱ is isomorphic to Z. Here $f^2=f$, though G has no identity element.

Theorem 2 makes use of the fact that if a partially ordered semigroup G has an integrally closed image by means of an m-congruence ϱ, then G has a group image by means of an m-congruence. However, G may have an integrally closed image G/ϱ which is not a group. An additional condition on $T(\varrho)$ necessary (and sufficient) for G/ϱ to be integrally closed is described in the following Theorem.

Theorem 3. *Let* G, ϱ, $T=T(\varrho)$ *be as in Theorem* 1. *Then* G/ϱ *is integrally closed if and only if*

(δ) *there exists* $f\in T$ *such that* $t.\cdot t=t^{\cdot}.t=f$ *for any* $t\in T$.

Proof. Suppose G/ϱ integrally closed, and let f be the maximum element in the identity class of G/ϱ. Then $f\varrho=t\varrho.\cdot t\varrho=(t.\cdot t)\varrho$ implies $f=t.\cdot t$; similarly $f=t^{\cdot}.t$. Conversely, let T satisfy (δ). Then

$$t\varrho.\cdot t\varrho=(t.\cdot t)\varrho=f\varrho=t\varrho^{\cdot}.t\varrho.$$

Since G/ϱ is self-residuated,

$$t\varrho=t\varrho.\cdot(t\varrho^{\cdot}.t\varrho)=t\varrho.\cdot f\varrho=t\varrho^{\cdot}.(t\varrho.\cdot t\varrho)=t\varrho^{\cdot}.f\varrho.$$

By Lemma 2, $f\varrho$ is the identity element of G/ϱ, whence G/ϱ is integrally closed.

In the example above, T satisfies (δ), for $f=(0, 0)\in T$ is such that $t:t=f$ for any $t\in T$. The semigroup $G/\varrho=U\times Z$ is integrally closed, but is not a group.

*

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