Transitivity of implication in orthomodular lattices

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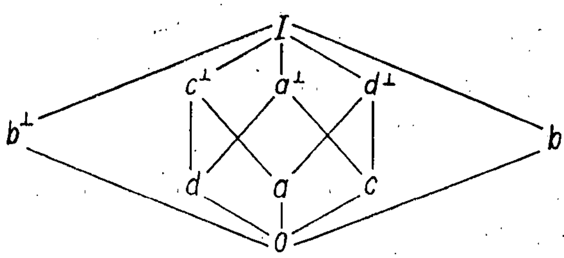
Extensive investigations in lattice theory [2—5] and logic [6—8] regarding the quantum logic of G. BIRKHOFF and J. VON NEUMANN [1] have induced many authors to call orthomodular lattices "generalized logic" [9]. KUNSEMÜLLER [6] pointed out that in quantum logic the relation of implication defined by

$$a^{\perp} \cup b = I$$

is not transitive ($a^{\perp}$ = orthocomplement of a, I = greatest element). The same holds also for orthomodular lattices in general. For example, the lattice with the diagram



is orthomodular and $a^{\perp} \cup b = I$, $b^{\perp} \cup c = I$, but $a^{\perp} \cup c \neq I$ in it.

KOTAS [7] has analysed the relations of implications defined on orthocomplemented modular lattices and characterized quantum logic on this basis by logical postulates. From the point of view of quantum logic the transitivity of the above established "classical" relation of implication is an interesting question. We have noticed that this property is characteristic to classical logic. Obviously it is the least we may demand of a logic L, and this we must demand [10], that L should be a lattice with unique orthocomplements. ROSE [11] has proved that such lattices coincide with orthomodular lattices.

We will prove that a lattice with unique orthocomplements is a Boolean algebra (i. e. a generalized logic is classical) if and only if the classical relation of implication defined in it is transitive.

Definition 1. A complemented lattice L is called *orthocomplemented* if the complementation is an involutory dualautomorphism in L, i. e. if to every $x \in L$ there exists an $x^{\perp} \in L$, such that

$$x \cap x^{\perp} = 0, \quad x \cup x^{\perp} = I, \quad x \leqq y \Rightarrow y^{\perp} \leqq x^{\perp}, \quad x^{\perp\perp} = x.$$

Definition 2. An element x of an orthocomplemented lattice L is said to be *orthogonal* to $y \in L$, in symbols $x \perp y$, if $x \leqq y^{\perp}$.

The relation of orthogonality is evidently symmetrical.

Definition 3. An orthocomplemented lattice L is called *uniquely orthocomplemented* if each $x \in L$ has at most one complement orthogonal to x.

Lemma 1. *The De Morgan laws hold in every orthocomplemented lattice L, i.e.*

$$(x \cap y)^{\perp} = x^{\perp} \cup y^{\perp} \quad \textit{and} \quad (x \cup y)^{\perp} = x^{\perp} \cap y^{\perp}$$

for each $x, y \in L$.

Proof. $x, y \leqq x \cup y \Rightarrow x^{\perp}, \quad y^{\perp} \geqq (x \cup y)^{\perp} \Rightarrow x^{\perp} \cap y^{\perp} \geqq (x \cup y)^{\perp}$

and

$$x^{\perp}, y^{\perp} \geqq x^{\perp} \cap y^{\perp} \Rightarrow x^{\perp\perp}, y^{\perp\perp} \leqq (x^{\perp} \cap y^{\perp})^{\perp} \Rightarrow x^{\perp\perp} \cup y^{\perp\perp} \leqq (x^{\perp} \cap y^{\perp})^{\perp} \Rightarrow$$

$$\Rightarrow x \cup y \leqq (x^{\perp} \cap y^{\perp})^{\perp} \Rightarrow (x^{\perp} \cap y^{\perp})^{\perp\perp} \leqq (x \cup y)^{\perp} \Rightarrow x^{\perp} \cap y^{\perp} \leqq (x \cup y)^{\perp},$$

whence $(x \cap y)^{\perp} = x^{\perp} \cup y^{\perp}$. The other statement follows by duality.

Lemma 2. *In every lattice L with unique orthocomplements we have*

$$a \leqq b \Rightarrow a = b \cap (b^{\perp} \cup a), \quad a, b \in L.$$

Proof. Let $a \leqq b$. Then $b^{\perp} \leqq a^{\perp}$ and according to Lemma 1 we get

$$a \cap [b \cap (b^{\perp} \cup a)]^{\perp} = a \cap [b^{\perp} \cup (b \cap a^{\perp})] \leqq a \cap [a^{\perp} \cup (b \cap a^{\perp})] = a \cap a^{\perp} = 0,$$

$$a \cup [b \cap (b^{\perp} \cup a)]^{\perp} = a \cup [b^{\perp} \cup (b \cap a^{\perp})] = (a \cup b^{\perp}) \cup (a^{\perp} \cap b)$$

$$= (a^{\perp} \cap b)^{\perp} \cup (a^{\perp} \cap b) = I,$$

$$[b \cap (b^{\perp} \cup a)]^{\perp} = b^{\perp} \cup (b \cap a^{\perp}) \leqq a^{\perp} \cup (b \cap a^{\perp}) = a^{\perp}.$$

Hence $[b \cap (b^{\perp} \cup a)]^{\perp}$ is a complement of a and also orthogonal to a, and so by the assumption is equal to $a^{\perp}$.

Lemma 3. *If L is a lattice with unique orthocomplements, then*

$$x \cap y = y \cap [(x \cup y)^{\perp} \cup x]$$

for each $x, y \in L$.

Proof. Applying Lemma 2 for $a = x$, $b = x \cup y$, we get

$$y \cap [(x \cup y)^{\perp} \cup x] = [y \cap (x \cup y)] \cap [(x \cup y)^{\perp} \cup x] =$$

$$= y \cap \{(x \cup y) \cap [(x \cup y)^{\perp} \cup x]\} = y \cap x.$$

Theorem. *Let L be a lattice with unique orthocomplements. If*

(1) $$(x^\perp \cup y = I, y^\perp \cup z = I) \Rightarrow x^\perp \cup z = I,$$

for each $x, y, z \in L$, then L is distributive (and thus is a Boolean algebra).

Proof. First we prove that (1) implies

(2) $$x^\perp \cup y = I \Rightarrow x \leqq y \quad \text{for each} \quad x, y \in L.$$

Let $x^\perp \cup y = I$, and
$$z = (x \cap y) \cup [x^\perp \cap (x \cup y)].$$
Then
$$y^\perp \cup z = y^\perp \cup \{(x \cap y) \cup [x^\perp \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{y^\perp \cup [x^\perp \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{y \cap [x \cup (x \cup y)^\perp]\}^\perp,$$
by Lemma 1, hence, by Lemma 3,
$$y^\perp \cup z = (x \cap y) \cup (x \cap y)^\perp = I.$$
Thus from (1) we obtain $x^\perp \cup z = I$, i. e.
$$I = x^\perp \cup z = x^\perp \cup \{(x \cap y) \cup [x^\perp \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{x^\perp \cup [x^\perp \cap (x \cup y)]\} = (x \cap y) \cup x^\perp.$$
But
$$(x \cap y) \cap x^\perp = (x \cap x^\perp) \cap y = 0 \cap y = 0,$$
and
$$(x \cap y) \leqq x = x^{\perp\perp}.$$
Consequently, $x \cap y$ is a complement of $x^\perp$ and is orthogonal to it, which means that
$$x \cap y = x^{\perp\perp} = x,$$
i. e.
$$x \leqq y.$$
Thus (2) is proved.

Let us take now $a, b \in L$ arbitrarily and let $x = b$, $y = a^\perp \cup (a \cap b)$. Then
$$x^\perp \cup y = I,$$
since
$$x^\perp \cup y = b^\perp \cup [a^\perp \cup (a \cap b)] = (b^\perp \cup a^\perp) \cup (a \cap b) = (a \cap b)^\perp \cup (a \cap b) = I.$$
Thus, according to (2), $x \leqq y$ which means

(3) $$b \leqq a^\perp \cup (a \cap b)$$

for each $a, b \in L$. Similarly, if $x = c$, $y = a^\perp \cup (a \cap c)$, where $a, c \in L$ are taken arbitrarily, then

(4) $$c \leqq a^\perp \cup (a \cap c).$$

By making use of (3) and (4), we get

$$a \cap (b \cup c) \leqq a \cap \{[a^{\perp} \cup (a \cap b)] \cup [a^{\perp} \cup (a \cap c)]\} =$$
$$= a \cap \{(a^{\perp} \cup a^{\perp}) \cup [(a \cap b) \cup (a \cap c)]\} =$$
$$= a \cap \{a^{\perp} \cup [(a \cap b) \cup (a \cap c)]\}.$$

Here, the last term is equal to $(a \cap b) \cup (a \cap c)$ by Lemma 2 (since $a \leqq (a \cap b) \cup (a \cap c)$). Therefore $a \cap (b \cup c) \leqq (a \cap b) \cup (a \cap c)$.

The opposite inequality, and the converse of the Theorem is well known. Cf. for instance [12].

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