Sums of operators with square zero*)

By P. A. FILLMORE in Bloomington (Indiana, U.S.A.)

Let $\mathfrak{H}$ be a separable infinite-dimensional complex Hilbert space. There are a number of results concerning generators in various senses for certain spaces of operators (bounded linear transformations) on $\mathfrak{H}$. For example, a von Neumann algebra is the linear span of its unitary elements [2, p. 4], the algebra of all operators on $\mathfrak{H}$ is generated as an algebra by its elements of square zero [3], and a von Neumann algebra with no abelian summand is generated as an algebra by its projections [5]. One of the most striking is the result of STAMPFLI [7] asserting that every operator on $\mathfrak{H}$ is the sum of eight idempotents. The purpose of this note is to show that STAMPFLI's theorem implies (in an elementary fashion) the following:

Theorem 1. *Every operator on $\mathfrak{H}$ is a sum of* 64 *operators with square zero.*

Theorem 2. *Every operator on $\mathfrak{H}$ is a linear combination of* 257 *projections.*

The author wishes to acknowledge many helpful and stimulating conversations with DAVID TOPPING, who also provided the proof of Lemma 3.

Several preliminary lemmas are necessary. An operator A is an *idempotent* if $A^2=A$, an (orthogonal) *projection* if $A^2=A$ and $A^*=A$, and an *involution* if $A^2=I$. We recall that P is an idempotent if and only if $2P-I$ is an involution. For any operator A with null-space $\mathfrak{N}=\text{null } A$ we write $\nu(A)=\dim \mathfrak{N}$. If the range $\mathfrak{R}=\text{ran } A$ is closed we write $\varrho(A)=\dim \mathfrak{R}$.

Lemma 1. *If P is an idempotent with $\nu(P)=\varrho(P)$ then the corresponding involution $S=2P-I$ is the sum of two operators with square zero.*

Proof. Since every idempotent is similar to a projection, the hypothesis implies easily that if $\mathfrak{K}$ is a separable infinite-dimensional Hilbert space, P is similar to the operator $\begin{pmatrix} I & O \\ O & O \end{pmatrix}$ on the space $\mathfrak{K}\oplus\mathfrak{K}$. Since $U=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is unitary, P is also similar to $U^*\begin{pmatrix} I & O \\ O & O \end{pmatrix}U=\frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$. Consequently, S is similar to $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$. But $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$ is the sum of $\begin{pmatrix} O & I \\ O & O \end{pmatrix}$ and $\begin{pmatrix} O & O \\ I & O \end{pmatrix}$, each having square zero, and the lemma follows.

*) Research supported in part by a grant from the National Science Foundation.

Lemma 2. *An idempotent P is either the sum or the difference of idempotents Q_1 and Q_2 such that* $\nu(Q_i)=\varrho(Q_i)$ $(i=1,2)$.

Proof. Suppose first that P is a projection. If $\varrho(P)=\infty$, then P is the sum of orthogonal projections Q_1 and Q_2 with $\varrho(Q_i)=\infty$; clearly then $\nu(Q_i)=\infty$. If $\varrho(P)<\infty$, then $\varrho(I-P)=\nu(P)=\infty$, so $I-P = Q_1+Q_2$ as above, and $P=(P+Q_1)-Q_1$ meets our requirements. Since any idempotent is similar to a projection, the lemma follows.

Proof of Theorem 1. Let A be an operator. By Stampfli's theorem and Lemma 2 we have $\frac{1}{2}A=\sum_{i=1}^{16}\pm P_i$, where each P_i is an idempotent with $\nu(P_i)=\varrho(P_i)$. But $P_i=\frac{1}{2}(S_i+I)$, where the involution S_i is the sum of two operators with square zero (by Lemma 1). Hence A is the sum of 32 operators with square zero and an integer multiple of I. Temporarily taking $A=\frac{1}{2}I$, we find that I is itself the sum of 32 operators with square zero. Consequently A is the sum of 64 operators with square zero.

Corollary 1. *Any operator is a sum of commutators.*

Proof.
$$\begin{pmatrix} O & A \\ O & O \end{pmatrix} = \begin{pmatrix} O & I \\ O & O \end{pmatrix}\begin{pmatrix} O & O \\ O & A \end{pmatrix} - \begin{pmatrix} O & O \\ O & A \end{pmatrix}\begin{pmatrix} O & I \\ O & O \end{pmatrix}.$$

This result was obtained by Halmos [6], and also follows from the recent description of commutators by Brown and Pearcy [1].

Corollary 2. *If K and L are any operators, there exist decompositions*
$$K = K_1K_1'+\ldots+K_nK_n' \quad \textit{and} \quad L = L_1L_1'+\ldots+L_nL_n' \qquad (n\leqq 64)$$
such that
$$K_i'K_i=L_i'L_i \qquad (i=1,\ldots,n).$$

Proof. To begin with, any operator on $\mathfrak{H}\oplus\mathfrak{H}$ of the form $\begin{pmatrix} AXE & AXF \\ CXE & CXF \end{pmatrix}$ has square zero, provided $EA+FC=O$. Conversely, any operator on $\mathfrak{H}\oplus\mathfrak{H}$ with square zero is similar to an operator $\begin{pmatrix} O & X \\ O & O \end{pmatrix}$, and consequently has the above form $\left(\text{multiply on the left by } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and on the right by its inverse } \begin{pmatrix} G & H \\ E & F \end{pmatrix}\right)$.

Applying Theorem 1 to the operator $\begin{pmatrix} K & O \\ O & -L \end{pmatrix}$, we find that $K=A_1X_1E_1+\ldots$ $\ldots+A_nX_nE_n$ and $L=-C_1X_1F_1-\ldots-C_nX_nF_n$ $(n\leqq 64)$ with $E_iA_i+F_iC_i=O$. Taking $L_i=A_i$, $L_i'=X_iE_i$, $K_i=-C_i$, and $K_i'=X_iF_i$, we have $L_i'L_i=X_iE_iA_i=$ $=-X_iF_iC_i=K_i'K_i$.

Lemma 3. *Any operator of the form* $\begin{pmatrix} O & A \\ O & O \end{pmatrix}$ *on* $\mathfrak{K}\oplus\mathfrak{K}$ *is a linear combination of 8 projections and I.*

Proof. The operator A is a linear combination of two self-adjoint contractions, each of which is a linear combination of two unitary operators [2, p. 4]. But

$$\begin{pmatrix} O & U \\ O & O \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix} + \frac{i}{2}\begin{pmatrix} I & -iU \\ iU^* & I \end{pmatrix} - \frac{1}{2}(1+i)I,$$

and the matrices on the right are easily seen to be multiples of projections when U is unitary.

Proof of Theorem 2. Let A be an operator with square zero. To apply Lemma 3 we need to know that A is unitarily equivalent to an operator of the form $\begin{pmatrix} O & B \\ O & O \end{pmatrix}$ on a space $\mathfrak{K} \oplus \mathfrak{K}$. The hypothesis implies that ran $A \subset$ null A and that $\nu(A) = \infty$. Therefore there exists a closed subspace $\mathfrak{K}$ between ran A and null A such that dim $\mathfrak{K}$ = dim $\mathfrak{K}^\perp$. Let U be a unitary operator from $\mathfrak{K}^\perp$ onto $\mathfrak{K}$, and define W from $\mathfrak{H}$ onto $\mathfrak{K} \oplus \mathfrak{K}$ by $Wx = y \oplus Uz$, where $x = y + z$ with $y \in \mathfrak{K}$ and $z \in \mathfrak{K}^\perp$. It is easy to see that W is unitary and that $WAW^* = \begin{pmatrix} O & B \\ O & O \end{pmatrix}$, where $B = AU^{-1}|\mathfrak{K}$. Lemma 3 now implies that any operator of square zero is a linear combination of 8 projections and I. But the proof of Theorem 1 shows that any operator is a sum of 32 operators of square zero and a multiple of I. Combining these statements completes the proof.

Remarks. 1. The numbers mentioned in the theorems are undoubtedly not the best possible, but we have not resolved this question.

2. Theorem 1 and both Corollaries fail if dim $\mathfrak{H} < \infty$, but Theorem 2 remains true. If, on the other hand, STAMPFLI's theorem is valid for nonseparable spaces (as seems likely), then so are the above results.

3. If either A or B is invertible, then AB and BA are similar. Thus the relation of Corollary 2 is an attenuated form of similarity.

4. The above results probably persist relative to a von Neumann algebra with no summand of finite type.

5. It is easy to see that a self-adjoint operator is a real linear combination of projections. However, it is not true that every positive operator is a linear combination of projections with positive coefficients. In fact, let A be positive and compact, and suppose $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with the $\lambda_i > 0$ and the P_i projections. Then $\lambda_i P_i \leqq A$ for each i, so that rng $P_i \subset$ rng $\sqrt{A}$ by [4]. Since $\sqrt{A}$ is compact, this implies that each P_i is finite-dimensional, and consequently so is A.

6. The real Banach space $\mathfrak{S}$ of all self-adjoint operators has the following curious property: it is the linear span of the extreme points of its unit ball $\mathfrak{U}$, but $\mathfrak{U}$ is not the convex hull of its extreme points. This is because $\mathfrak{U}$ is affinely equivalent to its positive part $\mathfrak{P}$ (by $U \to \frac{1}{2}(U+I)$), and the preceding remark shows that $\mathfrak{S}$ is the linear span of the extreme points of $\mathfrak{P}$, but that $\mathfrak{P}$ is not the convex hull of its extreme points.

References

[1] A. Brown and C. Pearcy, Structure of commutators of operators, *Ann. Math.*, **82** (1965) 112—127.
[2] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1957).
[3] R. G. Douglas and D. M. Topping, Operators whose squares are zero, *submitted.*
[4] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert, space, *Proc. Amer. Math. Soc.*, **17** (1966), 413—415.
[5] P. A. Fillmore and D. M. Topping, Operator algebras generated by projections, *submitted.*
[6] P. R. Halmos, Commutators of operators, *Amer. J. Math.*, **76** (1954), 191—198.
[7] J. G. Stampfli, Sums of projections, *Duke Math. J.*, **31** (1964), 455—462.

Indiana University, Bloomington, Indiana

(Received 15 August, 1966)