Positive definite contraction valued functions on locally compact abelian groups *)

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Introduction

In this paper we will discuss two separate yet related problems. In § 1 we ask under what conditions would the minimal unitary dilations of a positive definite contraction valued function on the LCA groups be unitary equivalent to the sum of many copies of the regular representation of the group. In § 2 we study the relationships between positive definiteness, von Neumann condition and Heinz condition for certain contraction valued functions in case of ordered groups. We are only interested in complex Hilbert spaces. We denote them by H, K, etc.; $B(H)$ (or $B(K)$) will be the algebra of bounded linear operators on H (or K). All topological spaces will be Hausdorff, and the notation $L^p(X, \Omega, \mu, C)$ for the set X, field Ω of subsets of X, measure μ and the complex field C will be as on p. 121 of [2].

§ 1

To study the first problem we make use of a new construction of the minimal unitary dilation of positive definite contraction valued function on LCA groups. For this purpose we need the following notations.

1. 1. Definition. Let $E(\cdot)$ be a bounded additive positive $B(H)$-valued set function defined on a field of subsets of a set X. If

$$f(x) = \sum_{l=1}^{r} \alpha_l \chi_{A_l}, \quad \Phi(x) = \sum_{i=1}^{m} h_i \chi_{D_i} \quad \text{and} \quad \Phi'(x) = \sum_{j=1}^{n} h'_j \chi_{D'_j}$$

are simple functions where α_l's are complex numbers, h_i, h'_j are in H and χ_ω denotes the characteristic function of the set ω in Ω, and A_l, D_i, D'_j are in Ω, then we define

$$\int_\omega f(x)(E(dx)\Phi(x), \Phi'(x)) = \sum_{i,j,l} \alpha_l (E(\omega \cap A_l \cap D_i \cap D'_j) h_i, h'_j)$$

whenever ω is in Ω. It is easily verified that this is independent of the representations

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of f, Φ, Φ'. Now suppose f is a bounded measurable complex valued function and Φ, Φ' are bounded measurable functions on X with values in a finite-dimensional subspace H_1 of H, choose a sequence f_n of simple complex valued functions converging to f uniformly on X, and sequences Φ_n, Φ_n' of simple measurable H_1-valued functions converging uniformly on X to Φ and Φ' respectively. For ω in Ω we then define

$$\int_\omega f(x)\big(E(dx)\Phi(x), \Phi'(x)\big) = \lim_{n\to\infty} \int_\omega f_n(x)\big(E(dx)\Phi_n(x), \Phi_n'(x)\big).$$

With standard argument it can be shown that this is independent of the choices of $H_1, f_n, \Phi_n, \Phi_n'$. Furthermore,

$$\int_\omega f(x)\big(E(dx)\Phi(x), \Phi'(x)\big) = \int_\omega \big(E(dx)f(x)\Phi(x), \Phi'(x)\big) =$$

$$= \int_\omega \big(E(dx)\Phi(x), \overline{f(x)}\,\Phi'(x)\big).$$

If we denote $\int_X \big(E(dx)\Phi(x), \Phi'(x)\big)$ by $\langle \Phi, \Phi' \rangle$ then $\langle \Phi, \Phi' \rangle$ is a positive definite Hermitian form.

Using the above notations we now give a new proof for LCA groups, to a theorem of SZ.-NAGY which we need later.

1.2. Theorem. *Every weakly continuous positive definite $B(H)$-valued function $\{T_\gamma\}$ on a LCA group Γ with $T_0 = I$ (0 is the identity of Γ) has a minimal unitary dilation $\{U_\gamma, K\}$.*

Proof. Let G be the dual group of Γ and $E(\cdot)$ the $B(H)$-valued set function on the Borel sets of G such that $T_\gamma = \int_G (x, \gamma)E(dx)$ [9]. Let D be the set of all H-valued bounded measurable functions with finite-dimensional range. If Φ, Ψ are in D, define $\langle \Phi, \Psi \rangle$ to be $\int_G \big(E(dx)\Phi(x), \Psi(x)\big)$. Thus D is a linear manifold with $\langle \Phi, \Psi \rangle$ as a positive definite scalar product (see Definition 1. 1). Denote by N the linear subspace of D consisting of those Φ for which $\langle \Phi, \Phi \rangle = 0$. Denote D/N by K_0 and the coset $\Phi + N$ in K_0 by $[\Phi]$. Then $\langle [\Phi], [\Psi] \rangle = \langle \Phi, \Psi \rangle$ is well-defined on K_0 so that K_0 is an inner product space and its completion K is a Hilbert space. Define U_γ on K_0 by $U_\gamma[\Phi] = [\Psi]$ where $\Psi(x) = (x, \gamma)\Phi(x)$. It is easily verified that the map is independent of the choice of coset representatives and is in fact an isometry of K_0 onto itself. Thus U_γ extends by continuity to a unitary transformation of K (which we also denote by U_γ). Evidently $\{U_\gamma\}$ is a unitary representation of Γ. Given any two elements Φ, Ψ in D let H_1 be the finite-dimensional subspace of H generated by the ranges of Φ and Ψ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of H_1. Since $\big(E(\cdot)e_i, e_j\big)$ is a finite regular Borel measure, so we have

$$\int_G (x, \gamma)\big(E(dx)\Phi(x), \Psi(x)\big) = \sum_{i,j=1}^{n} \int_G (x, \gamma)\big(\Phi(x), e_i\big)\overline{\big(\Psi(x), e_j\big)}\big(E(dx)e_i, e_j\big)$$

which is a continuous function of γ for $[\Phi]$, $[\Psi]$ in K_0. By the uniform boundedness of $\{U_\gamma\}$, $\{U_\gamma\}$ is weakly continuous on Γ. Embed H in K by mapping h in H to $[\Phi_h]$ in K_0 where $\Phi_h(x)=h$ for all x in G. Obviously this is a linear and isometric embedding. For arbitrary h, h' in H we have

$$(U_\gamma[\Phi_h], [\Phi_{h'}]) = \int_G (x, \gamma)(E(dx)h, h') = (T_\gamma h, h')$$

so $PU_\gamma P = T_\gamma$ where P is the projection from K onto H. Finally a standard argument would show that the elements of the form $U_\gamma[\Phi_h]$ where h in H generate K, so $\{U_\gamma, K\}$ is a minimal unitary dilation.

For the remainder of this section we shall assume H is separable. If Γ is σ-compact (in particular if Γ is the integer group or the group of real numbers) this is only a slight restriction since in this case H is an orthogonal direct sum of separable subspaces each of which is invariant under T_γ for all γ in Γ as we shall see later (Remark 1. 8).

Suppose that H is separable. Denote the dual group of Γ by G and the Haar measure of G by σ. Suppose there exists a positive $B(H)$-valued function $M(\cdot)$ on G such that for any Borel set ω of G and any h, h' in H we have

$$(E(\omega)h, h') = \int_\omega (M(x)h, h')\sigma(dx).$$

Set $H(x)=\overline{M(x)H}=\overline{M(x)^{1/2}H}$. Then $x\to H(x)$ is a field of Hilbert spaces on G. Define the unitary operator S_γ on the direct integral space

$$\mathbf{H} = \int^{\oplus} H(x)\sigma(dx)$$

by $(S_\gamma\xi)(x)=(x, \gamma)\xi(x)$ where ξ is in $\mathbf{H}$. We first establish

1. 3. Theorem. *$\{\mathbf{H}, S_\gamma\}$ is unitarily equivalent to the minimal unitary dilation $\{U_\gamma, K\}$ of $\{T_\gamma, H\}$.*

Proof. We use the same notations as in the proof of theorem 1. 2. Define W_0 on D into $\mathbf{H}$ by $(W_0\Phi)(x)=M(x)^{1/2}\Phi(x)$ for Φ in D. It is easy to verify that

$$\int_G \|M(x)^{1/2}\Phi(x)\|^2\sigma(dx) = \langle\Phi, \Phi\rangle$$

so W_0 is a linear isometry map from D into $\mathbf{H}$. We claim that the range of W_0 is dense in $\mathbf{H}$. Let $\{g_i(\cdot)|\, i=1, 2, \ldots\}$ be a measurable field of orthonormal bases, so

$$g_i(x) = \sum_{j=1}^{m_i} c_j^i(x)\big(M(x)^{1/2}e_j\big)$$

where $c_j^i(x)$ are complex valued measurable functions. Suppose ξ in $\mathbf{H}$ and $\varepsilon>0$ are given. Then $\xi(x)=\sum_{i=1}^{k}\alpha_i(x)g_i(x)$ where $\alpha_i(x)=(\xi(x), g_i(x))$ is measurable.

By the monotone convergence theorem we can find an integer k such that if we define $\xi_1(x)=\sum_{i=1}^{\infty}\alpha_i(x)g_i(x)$ then

$$\int_G \|\xi(x)-\xi_1(x)\|^2\sigma(dx) = \int_G\left(\sum_{i=k+1}^{\infty}|\alpha_i(x)|^2\right)\sigma(dx) < \varepsilon^2/4$$

or $\|\xi-\xi_1\|<\varepsilon/2$. A similar argument shows that there is a positive constant C such that if we define $\xi_2(x)=\xi_1(x)$ whenever $|c_j^i(x)|\leqq C$ and $|\alpha_i(x)|\leqq C$ for $j\leqq m_i$ and $i\leqq k$ and we define $\xi_2(x)=0$ otherwise, then $\|\xi_2-\xi_1\|<\varepsilon/2$ so $\|\xi_2-\xi\|<\varepsilon$. Now define $\eta(x)=\sum_{i=1}^{k}\alpha_i(x)\cdot\sum_{j=1}^{m_i}c_j^i(x)e_i$ if $\xi_2(x)\neq 0$ and $\eta(x)=0$ if $\xi_2(x)=0$. It follows easily that η is in D and $W_0\eta=\xi_2$ so W_0D is dense in $\mathbf{H}$. Now W_0 induces an isometry of K_0 onto a dense subspace of $\mathbf{H}$ which extends by continuity to a unitary map W of K onto $\mathbf{H}$ and clearly $WU_\gamma=S_\gamma W$ so W is a unitary equivalence between two representations of Γ.

Now we are going to answer partly the first problem mentioned earlier.

1.4. Theorem. *Under the hypotheses of* 1.3, *the minimal unitary dilation of* $\{T_\gamma, H\}$ *is unitarily equivalent to the sum of* d_0 *copies of the regular representation of* Γ *iff* $\dim(H(x))=d_0$ *for almost all* x *with respect to* σ.

Proof. Assume that $\dim(H(x))=d_0$ for almost all x. Without loss of generality we may assume $\dim(H(x))=d_0$ for all x. Let $\{g_i(\cdot)\,|\,i=1,2,\ldots\}$ be a measurable field of orthonormal bases for $H(x)$. We map ξ in $\mathbf{H}$ to the element $(\alpha_1,\alpha_2,\ldots)$ in the direct sum of d_0 copies of $L^2(G,\Omega,\sigma,C)$ such that $\alpha_i(x)=(\xi(x),g_i(x))$. It can be verified that this gives a unitary equivalence between $\{\mathbf{H},S_\gamma\}$ and the sum of d_0 copies of regular representation of Γ. By 1.3 it now follows that the minimal unitary dilation of $\{T_\gamma,H\}$ is unitarily equivalent to the sum of d_0 copies of regular representation of Γ. For the converse now assume that the minimal unitary dilation of $\{T_\gamma,H\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of Γ. Since H is assumed to be separable, if $\{U_\gamma,K\}$ denotes the minimal unitary dilation it follows that K is countably generated. That is there is a countable subset of K such that the closed subspace invariant under all U_γ generated by this countable set is the whole K. Therefore the regular representation of Γ is countably generated and thus G is σ-compact and its Haar measure σ is σ-finite. For a measurable set ω in G define $A(\omega)$ on $\mathbf{H}$ by $A(\omega)\xi(x)=\chi_\omega(x)\xi(x)$. ($\chi_\omega$ is the characteristic function of ω.) It is readily seen that $A(\cdot)$ is the spectral family given by STONE's theorem for the representation $\{S_\gamma\}$ of Γ, so $A(\omega)$ is in the weakly closed algebra of operators generated by the $\{S_\gamma\}$. Now G may be decomposed into disjoint measurable sets ω_p for $p=1,2,\ldots$ such that $\dim(H(x))=p$ if x is in ω_p. Denote $A(\omega_p)$ by A_p. Define the representation $\{V_{\gamma,p}\}$ as follows: $V_{\gamma,p}$ acts in the space $L^2(\omega_p,\Omega_p,\sigma,C)$ (where Ω_p is the σ-field of Borel subsets of ω_p) and is such that $(V_{\gamma,p}\zeta)(x)=(x,\gamma)\zeta(x)$. The argument used in the first part of the proof will also show that the representation $\{A_pS_\gamma A_p\}$ on the range of A_p is unitarily equivalent to the sum of p copies of the regular representation $\{V_{\gamma,p}\}$. Now since G is σ-finite, so is ω_p, hence there exists a nowhere vanishing L^2 function on ω_p and we conclude that the representation $\{V_{\gamma,p}\}$ is cyclic, and thus locally simple (see p. 42 of [3]). Thus corresponding to the decomposition

$$I = A_\infty + A_1 + A_2 + \dots$$

we obtain a decomposition

$$\{S_\gamma\} = \infty\{V_{\gamma,\infty}\} + 1\{V_{\gamma,1}\} + 2\{V_{\gamma,2}\} + \dots .$$

Since the A_i's are in the weak closure of the $\{S_\gamma\}$, they are in the center of the algebra of intertwining operators, so this decomposition is the unique decomposition of $\{S_\gamma\}$ (see p. 40 of [3]). It follows that $\{S_\gamma\}$ is unitarily equivalent to the sum of d_0 copies of a locally simple representation only if the only non-trivial term in this decomposition is the one for $p = d_0$. Thus $\dim(H(x)) = d_0$ for almost all x.

The following results are the applications of this theorem to some special cases. The first is a result of Sz.-Nagy and Foiaş (see p. 125 of [8]) but we give a different proof here.

1. 5. Theorem. *Let T be a completely non-unitary contraction of H and suppose the intersection of the spectrum of T with the unit circle is a set of measure zero. Define $T_n = T^n$ for $n \geq 0$ and $T_n = T^{*(-n)}$ for $n < 0$ and let $\{U_n\}$ be its minimal unitary dilation. Then $\{U_n\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of the integer group where $d_0 = \dim \overline{(I - T^*T)H} = \dim \overline{(I - TT^*)H}$.*

Proof. Denote the intersection of the spectrum of T with the unit circle by ω_0. Since ω_0 has Lebesgue measure zero, by Theorem 2 of [7] we have $E(\omega_0) = O$ where $E(\cdot)$ is the $B(H)$-valued set function on the circle such that

$$T_n = \int_0^{2\pi} e^{in\theta} E(d\theta).$$

For $z = re^{i\theta}$ $(r < 1)$ define

$$M(z) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{-im\theta} T_m = \operatorname{Re}[(I + \bar{z}T)(I - \bar{z}T)^{-1}].$$

It is obvious that $(M(z)h, h')$ is the Poisson integral of the measure $(E(\cdot)h, h')$ for h, h' in H. If F is any closed interval on the unit circle which does not intersect ω_0, we notice that $M(z)$ has a harmonic extension $\operatorname{Re}[(I + \bar{z}T)(I - \bar{z}T)^{-1}]$ to a neighborhood of F, denote this extension by $\tilde{M}(z)$. Using Fatou's theorem it then follows that if ω is any measurable subset of F,

$$(E(\omega)h, h') = \int_\omega (\tilde{M}(z)h, h')\sigma(dz) \text{ for } h, h' \text{ in } H.$$

If we extend $\tilde{M}(z)$ to the whole circle by defining $\tilde{M}(z) = O$ if z is in ω_0, then from the fact that $E(\omega_0) = O$ it follows that $(E(\omega)h, h') = \int_\omega (M(z)h, h')\alpha(dz)$ for any measurable set ω on the circle. Now for z on the circle but not in ω_0, we have

$$\tilde{M}(z) = (I - zT^*)^{-1}(I - T^*T)(I - \bar{z}T)^{-1} = (I - \bar{z}T)^{-1}(I - TT^*)(I - zT^*)^{-1}.$$

Thus $\dim \overline{\tilde{M}(z)H} = \dim \overline{(I - T^*T)H} = \dim \overline{(I - TT^*)H} = d_0$ for almost all z in the circle group. Using theorem 1. 4 we then have the desired conclusion.

1. 6. Theorem. *Let Γ be the group of real numbers and $\{T_t\}$ a weakly continuous positive definite $B(H)$-valued function on Γ with $T_0 = I$. Assume that $\limsup_{t\to\infty} \|T_t\|^{1/t} < 1$. Then the minimal unitary dilation $\{U_t\}$ of $\{T_t\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of Γ where $d_0 \leqq \dim H$.*

Proof. Define $M(z)$ by

$$(M(z)h, h') = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (e^{-izt} T_t h, h')\, dt.$$

Then for

$$|\operatorname{Im} z| < -\log \left(\limsup_{t\to+\infty} \|T_t\|^{1/t}\right)$$

$M(z)$ exists and is a bounded operator, a positive operator if z is a real number and $(M(z)h, h')$ is an analytic function for every h, h' in H. If G is the dual group of Γ (that is G is the group of real numbers) and if $E(\cdot)$ is the $B(H)$-valued set function on G such that $T_t = \int_{-\infty}^{\infty} e^{ist} E(ds)$, from the inversion theorem on Fourier transform we have $(E(\omega)h, h') = \int_{\omega} \frac{1}{\sqrt{2\pi}} (M(s)h, h')ds$ for any Borel set ω of G and h, h' in H. We claim that $\dim \overline{M(s)H}$ is a constant for almost all real numbers with respect to Lebesgue measure. In fact let s_0 be a real number and n an integer such that $\dim \overline{M(s_0)H} \geqq n$. Select elements $h_1, h_2, \ldots, h_n$ in H so that $M(s_0)h_1$, $M(s_0)h_2, \ldots, M(s_0)h_n$ are linearly independent vectors. Consider the Gram determinant of $M(s)h_1, M(s)h_2, \ldots, M(s)h_n$. This is a real analytic function of s which does not vanish at s_0, hence it vanishes for at most a countable set. Thus $M(s)h_1, \ldots$ $\ldots, M(s)h_n$ are linearly independent except for at most countably many values of s. From the separability of H it follows that $\dim \overline{M(s)H}$ is constant almost everywhere, and the conclusion follows from Theorem 1. 4.

Instead of analytic functions on the real line we may consider analytic functions on the circle. So we get

1. 7. Theorem. *Let Γ be the group of integers and $\{T_n\}$ a positive definite $B(H)$-valued function on Γ such that $T_0 = I$. Assume that $\limsup \|T_n\|^{1/n} < 1$. Then the unitary dilation of $\{T_n\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of the integer group.*

1. 8. Remark. If Γ is σ-compact, many of above discussions can be applied for non-separable H. If $\{T_\gamma\}$ is positive definite with $\{U_\gamma\}$ its minimal unitary dilation, observe that since $\{U_\gamma\}$ is strongly continuous, so is $\{T_\gamma\}$. Thus if h is an element of H, the set of $\{T_\gamma h \mid \gamma \in \Gamma\}$ is a σ-compact subset of the metric space H, hence separable and so generate a separable subspace. From this we conclude by a standard argument that the smallest subspace containing h and invariant under all T_γ is separable. Using Zorn's Lemma and standard argument we can show that H may be written as a direct sum of subspaces H_α each of which is separable and invariant under all T_γ. If $T_{\gamma,\alpha}$ denotes the restriction of T_γ to H_α and if $\{U_{\gamma,\alpha}\}$ is the minimal unitary dilation of $\{T_{\gamma,\alpha}\}$, it is easily verified that the minimal uni-

tary dilation of $\{T_\gamma\}$ is the direct sum of $\{U_{\gamma,\alpha}\}$. Thus the results obtained in separable case are also true for general H provided that Γ is σ-compact. In particular, 1. 5, 1. 6, 1. 7 can be extended to the non-separable case in this manner.

§ 2

In 1951 VON NEUMANN proved that if T is a contraction, and for every analytic function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty}|a_n|<\infty$ and $|f(z)|\leqq 1$ for $|z|\leqq 1$, we define $f(T)$ to be $\sum_{n=0}^{\infty} a_n T^n$, then we have $\|f(T)\|\leqq 1$. Later E. HEINZ proved that if instead of $|f(z)|\leqq 1$, we require that $\operatorname{Re} f(z)\geqq 0$ for $|z|\leqq 1$, then for any h in H we have $\operatorname{Re}(f(T)h, h)\geqq 0$. From this result HEINZ could give an easy proof of the von Neumann theorem. In [6] SZ.-NAGY showed that the von Neumann and Heinz theorems follow easily from the existence of unitary dilations. Here we shall exhibit the relationship between the positive definiteness, the von Neumann theorem and the Heinz theorem in a more general setting. As both theorems depend on positive elements of the integer group, we shall only consider ordered LCA groups from now on. If Γ is an ordered group, Γ^+ will denote the set of all non-negative elements of Γ, Σ the σ-field of Borel sets of Γ and ϱ the Haar measure of Γ.

2. 1. Definition. Let A_1 denote the set of all functions ξ in $L^1(\Gamma, \Sigma, \varrho, \mathbf{C})$ such that $\xi(\gamma)=0$ for $\gamma<0$. With the L^1-norm and the convolution as multiplication A_1 is a Banach algebra. (See p. 380 of [1].) Define a norm $|||\ |||$ on A_1 by $|||\xi|||=\|\hat{\xi}\|_\infty$ where $\hat{\xi}$ is the Fourier transform of ξ in A_1 and $\|\ \|_\infty$ is the sup norm. Denote the completion of A_1 in the norm $|||\ |||$ by A_0.

2. 2. Definition. Let $\mathscr{T}=\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. For ξ in A_1 define $\xi(\{T_\gamma\})$ to be $\int_\Gamma \xi(\gamma)T_\gamma\,\varrho(d\gamma)$ where the integral is taken in the weak sense. It is clear that $\|\xi(\{T_\gamma\})\|\leqq\|\xi\|_1$.

2. 3. Remark. Let $\{T_\gamma\}$ be the same as in 2. 2 and in addition positive definite. Let $E(\cdot)$ be the $B(H)$-valued set function defined on the dual group G of Γ such that $T_\gamma=\int_G (x,\gamma)E(dx)$. For any bounded Borel measurable complex valued function φ on G, $\int_G \varphi(x)E(dx)$ defines an element of $B(H)$ which will be denoted by $\varphi(\mathscr{T})$. If $\{U_\gamma, K\}$ is the minimal unitary dilation of $\{T_\gamma, H\}$ and $F(\cdot)$ is the spectral measure of $\{U_\gamma\}$, then the map from φ to $\varphi(\mathscr{U})=\int_G \varphi(x)F(dx)$ is a homomorphism of the B^*-algebra of bounded Borel functions on G (under pointwise multiplication) into the B^*-algebra $B(K)$. If P is the projection from K onto H, so that $PU_\gamma P=T_\gamma$ and $PF(\cdot)P=E(\cdot)$, then $\varphi(\mathscr{T})=P\varphi(\mathscr{U})P$. If ξ is in A_1, $\hat{\xi}$ is complex valued bounded measurable (in fact continuous) so $\hat{\xi}(\mathscr{T})=\int_G \hat{\xi}(x)E(dx)$ is defined. From FUBINI's theorem it follows that $\hat{\xi}(\mathscr{T})=\xi(\{T_\gamma\})$.

2. 4. Definition. Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. We say that $\{T_\gamma\}$ satisfies the von Neumann condition if ξ in A_1 implies $\|\xi(\{T_\gamma\})\|\leqq|||\xi|||=\|\hat{\xi}\|_\infty$. We say that $\{T_\gamma\}$ satisfies the Heinz condition if ξ in A_1 and $\operatorname{Re}\hat{\xi}(x)\geqq 0$ for all x in G imply that $\operatorname{Re}(\xi(\{T_\gamma\})h, h)\geqq 0$ for all h in H.

2. 5. Proposition. *If $\{T_\gamma\}$ is a $B(H)$-valued weakly continuous positive definite function on Γ with $T_0=I$, then $\{T_\gamma\}$ satisfies both the von Neumann condition and the Heinz condition.*

Proof. Let $E(\cdot)$ be the $B(H)$-valued set function on the dual group G such that $T_\gamma=\int_G(x,\gamma)E(dx)$. Suppose $\{U_\gamma, K\}$ is the minimal unitary dilation of $\{T_\gamma, H\}$ and $F(\cdot)$ is the spectral measure of $\{U_\gamma\}$. For an arbitrary element ξ in A_1, we have $\left\|\int_G\hat{\xi}(x)F(dx)\right\|\leqq\|\hat{\xi}\|_\infty$ (see p. 900 of [2]). Thus

$$\left\|\int_G\hat{\xi}(x)E(dx)\right\|\leqq\|\hat{\xi}\|_\infty,\quad\text{so}\quad\|\xi(\{T_\gamma\})\|\leqq|||\xi|||$$

(see 2. 3). Hence $\{T_\gamma\}$ satisfies the von Neumann condition. Next suppose that $\operatorname{Re}\hat{\xi}(x)\geqq 0$ for all x in G. For h in H we have

$$\operatorname{Re}(\xi(\{T_\gamma\})h, h)=\operatorname{Re}\int_G\hat{\xi}(x)(E(dx)h, h)=\int_G\operatorname{Re}\hat{\xi}(x)(E(dx)h, h)$$

which is non-negative because $\operatorname{Re}\hat{\xi}(x)\geqq 0$ and $E(\omega)$ is a positive operator for every ω in Ω (σ-field of Borel subsets of G). Thus $\{T_\gamma\}$ satisfies the Heinz condition.

2. 6. Proposition. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. If $\{T_\gamma\}$ satifies the Heinz condition, $\{T_\gamma\}$ is positive definite.*

Proof. For ζ in $L^1(\Gamma, \Sigma, \varrho, C)$ define $\tilde{\zeta}$ by $\tilde{\zeta}(\gamma)=\overline{\zeta(-\gamma)}$ for γ in Γ. It is easily verified that $(\zeta*\tilde{\zeta})(\gamma)=\overline{(\zeta*\tilde{\zeta})(-\gamma)}$. Define ξ by

$\xi(\gamma)=2(\zeta*\tilde{\zeta})(\gamma)$ if $\gamma>0$,

$\xi(\gamma)=0$ if $\gamma<0$, and

$\xi(0)=(\zeta*\tilde{\zeta})(0)$.

It is readily seen that ξ is in A_1 and for x in G, $\operatorname{Re}\hat{\xi}(x)=\hat{\zeta}(x)\hat{\tilde{\zeta}}(x)=|\hat{\zeta}(x)|^2\geqq 0$. Using Heinz condition we get $\operatorname{Re}(\xi(\{T_\gamma\})h, h)\geqq 0$ for every h in H. Thus $(\operatorname{Re}\xi(\{T_\gamma\})h, h)\geqq 0$ where

$$\operatorname{Re}\xi(\{T_\gamma\})=1/2[\xi(\{T_\gamma\})+\xi(\{T_\gamma\})^*]=$$

$$=1/2\int_\Gamma[\overline{\xi(-\gamma)}+\xi(\gamma)]T_\gamma\varrho(d\gamma)=\int_\Gamma(\zeta*\tilde{\zeta})(\gamma)T_\gamma\varrho(d\gamma).$$

(All integrals are taken in the weak sense.) Therefore $\int_\Gamma(\zeta*\tilde{\zeta})(\gamma)(T_\gamma h, h)\varrho(d\gamma)\geqq 0$, that is the bounded continuous function $(T_\gamma h, h)$ on Γ is an integral positive definite function and so equal locally almost everywhere to a continuous positive definite function (see p. 397 of [4]). Thus from the continuity of $(T_\gamma h, h)$ it follows that it is positive definite [9]. Hence $\{T_\gamma\}$ is positive definite.

2. 7. Proposition. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ with $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. If $\{T_\gamma\}$ satisfies the von Neumann condition, $\{T_\gamma\}$ is positive definite.*

Proof. Denote by α the map from ξ in A_1 to $\xi(\{T_\gamma\})$. Clearly α is linear and norm decreasing. Since $\{T_\gamma\}$ satisfies the von Neumann condition $\|\alpha(\xi)\| \leqq |||\xi||| = \|\hat{\xi}\|_\infty$. Thus α extends by continuity to A_0. For h in H we define the linear functional $\tilde{h}$ on A_0 by $\tilde{h}(\eta)=(\alpha(\eta)h, h)$ for η in A_0. Since α is norm decreasing, $\|\tilde{h}\| \leqq \|h\|^2$. Thus by the Hahn—Banach and Riesz representation theorems there is a measure μ_h on the Borel sets of G such that for ξ in A_1, $\tilde{h}(\xi)=\int_G \hat{\xi}(x)\mu_h(dx)$ and μ_h has total variation at most $\|h\|^2$. We claim that $\mu_h(G)=\|h\|^2$ from which it will follow that μ_h is a positive measure. Since Γ is ordered, it is metric (see p. 196 of [5]) and thus satisfies the first axiom of countability. Choose a decreasing sequence $\{N_k\}$ of compact neighborhoods of 0 in Γ which form a neighborhoods base at 0. Define φ_k by

$$\varphi_k(\gamma)=1/\varrho(N_k\cap\Gamma^+) \quad \text{if} \quad \gamma\in N_k\cap\Gamma^+,$$

$$\varphi_k(\gamma)=0 \quad \text{otherwise}.$$

Clearly $\varphi_k\in A_1$ and $\|\varphi_k\|=1$. Furthermore it is readily verified that $\hat{\varphi}_k(x)\to 1$ for every x in G as $k\to\infty$. Thus by the dominated convergence theorem,

$$\lim_{k\to\infty}\int_G \hat{\varphi}_k(x)\mu_h(dx) = \mu_h(G).$$

On the other hand,

$$\lim_{k\to\infty}\int_G \hat{\varphi}_k(x)\mu_h(dx) = \lim_{k\to\infty}\int_\Gamma \varphi_k(\gamma)(T_\gamma h, h)\varrho(d\gamma) = \|h\|^2.$$

Thus μ_h is a positive measure. Since it is clear that for ξ in A_1,

$$\int_\Gamma \xi(\gamma)(T_\gamma h, h)\varrho(d\gamma) = \int_\Gamma \xi(\gamma)\hat{\mu}_h(\gamma)\varrho(d\gamma)$$

so we conclude from the continuity of $(T_\gamma h, h)$ and $\hat{\mu}_h(\gamma)$ that they are equal for γ in Γ^+. Since $T_{-\gamma}=T_\gamma^*$ and $\overline{\hat{\mu}_h(-\gamma)}=\hat{\mu}_h(\gamma)$ it follows that $(T_\gamma h, h)=\hat{\mu}_h(\gamma)$ for all γ in Γ. Thus $(T_\gamma h, h)$ is the Fourier—Stieltjes transform of the positive measure μ_h, so is positive definite, that is, $\{T_\gamma\}$ itself is positive definite.

2. 8. Theorem. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. Then the following three statements are equivalent:* (1) *$\{T_\gamma\}$ is positive definite.* (2) *$\{T_\gamma\}$ satisfies the von Neumann condition.* (3) *$\{T_\gamma\}$ satisfies the Heinz condition.*

Proof. Propositions 2. 5, 2. 6, 2. 7.

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