On the spectrum of unitary ϱ-dilations

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We shall consider operators T in a Hilbert space $\mathfrak{H}$, which admit *unitary ϱ-dilations* ($\varrho>0$), i. e. unitary operators U defined on some Hilbert space $\mathfrak{K}(\supset\mathfrak{H})$, such that

$$T^n\varphi=\varrho PU^n\varphi \qquad (\varphi\in\mathfrak{H};\ n=1,2,\ldots), \tag{1}$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Obviously, (1) implies

$$T^{*n}\varphi=\varrho PU^{-n}\varphi. \tag{1*}$$

B. SZ.-NAGY and C. FOIAŞ have characterized the operators T which have unitary ϱ-dilations; see [4].

We shall denote by $\sigma(U)$ the spectrum of U. In the case $\varrho=1$, $\sigma(U)$ has been studied extensively, cf. in particular [1], [2], [3]. The purpose of this paper is to study the spectral properties of U for arbitrary $\varrho>0$.

First we recall some definitions, familiar in the case $\varrho=1$, but which extend immediately to the general case too.

The unitary ϱ-dilation U of T is *minimal*, if

$$\mathfrak{K}=\bigvee_{n=-\infty}^{\infty}U^n\mathfrak{H}. \tag{2}$$

In this case U is uniquely determined up to isomorphism. (The proof is similar to that given in the special case $\varrho=1$, cf. [$*$].)

Let E_ϑ ($0\leqq\vartheta\leqq2\pi$) be the spectral function of U. We say that the spectral measure of U is *absolutely continuous* if, for every vectors $\varphi,\psi\in\mathfrak{K}$, the function $(E_\vartheta\varphi,\psi)$ of ϑ is absolutely continuous on $0\leqq\vartheta\leqq2\pi$, i. e., if there exists a function $f_{\varphi,\psi}(\vartheta)\in L(0,2\pi)$ such that

$$(E_\vartheta\varphi,\psi)=\int_0^\vartheta f_{\varphi,\psi}(\tau)\,d\tau. \tag{3}$$

T is called completely non-unitary, if there exists no vector $\varphi\in\mathfrak{H}$, $\varphi\neq0$, for which

$$\ldots=\|T^{*2}\varphi\|=\|T^*\varphi\|=\|\varphi\|=\|T\varphi\|=\|T^2\varphi\|=\ldots$$

We shall use the following notations:

$$\mathfrak{L}_n=U^n\overline{(U-T)\mathfrak{H}},\quad \mathfrak{L}_n^*=U^{*n}\overline{(U^*-T^*)\mathfrak{H}} \qquad (n=0,\pm1,\ldots), \tag{4}$$

$$\mathfrak{L}=\bigvee_{n=-\infty}^{\infty}\mathfrak{L}_n,\quad \mathfrak{L}^*=\bigvee_{n=-\infty}^{\infty}\mathfrak{L}_n^*. \tag{5}$$

We denote by Q the orthogonal projection of $\mathfrak{L}$ onto $\bigvee_{n=1}^{\infty} \mathfrak{L}_n$, and by Q' the orthogonal projection of $\mathfrak{L}^*$ onto $\bigvee_{n=1}^{\infty} \mathfrak{L}_n^*$. Further, we set

$$\mathfrak{V} = \overline{Q\mathfrak{L}_0}, \quad \mathfrak{V}^* = \overline{Q'\mathfrak{L}_0^*}, \tag{6}$$

$$\mathfrak{W} = \overline{(I-Q)\mathfrak{L}_0}, \quad \mathfrak{W}^* = \overline{(I-Q')\mathfrak{L}_0^*}. \tag{7}$$

Lemma 1. *$\mathfrak{L}_n \perp \mathfrak{L}_k$ and $\mathfrak{L}_n^* \perp \mathfrak{L}_k^*$ if $n-k \geqq 2$.*

Proof. (4) shows that it sufficies to prove

$$\left.\begin{array}{ll} (8) & (U^m(U-T)\varphi, (U-T)\psi) = 0 \\ (8^*) & (U^{*m}(U^*-T^*)\varphi, (U^*-T^*)\psi) = 0 \end{array}\right\} \quad \text{if} \quad m = 2, 3, \dots,$$

where φ and ψ are arbitrary vectors in $\mathfrak{H}$.

In order to prove (8), we use (1):

$$(U^m(U-T)\varphi, (U-T)\psi) = (U^m\varphi, \psi) - (U^{m-1}T\varphi, \psi) - (U^{m+1}\varphi, T\psi) + (U^mT\varphi, T\psi) =$$

$$= \left(\frac{1}{\varrho}T^m\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{m-1}T\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{m+1}\varphi, T\psi\right) + \left(\frac{1}{\varrho}T^mT\varphi, T\psi\right) = 0.$$

Similarly, using (1*) we get (8*) as follows:

$$(U^{*m}(U^*-T^*)\varphi, (U^*-T^*)\psi) =$$

$$= (U^{-m}\varphi, \psi) - (U^{-m+1}T^*\varphi, \psi) - (U^{-m-1}\varphi, T^*\psi) + (U^{-m}T^*\varphi, T^*\psi) =$$

$$= \left(\frac{1}{\varrho}T^{*m}\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{*m-1}T^*\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{*m+1}\varphi, T^*\psi\right) + \left(\frac{1}{\varrho}T^{*m}T^*\varphi, T^*\psi\right) = 0.$$

Lemma 2. *$\mathfrak{V}$, $\mathfrak{V}^*$, $\mathfrak{W}$ and $\mathfrak{W}^*$ are wandering subspaces for U. I. e., if $n \neq k$ then*

$$U^n\mathfrak{V} \perp U^k\mathfrak{V}, \quad U^n\mathfrak{V}^* \perp U^k\mathfrak{V}^*,$$

$$U^n\mathfrak{W} \perp U^k\mathfrak{W}, \quad U^n\mathfrak{W}^* \perp U^k\mathfrak{W}^*.$$

Proof. It suffices to prove that

$$U^mQ\mathfrak{L}_0 \perp Q\mathfrak{L}_0, \quad U^mQ'\mathfrak{L}_0^* \perp Q'\mathfrak{L}_0^*, \tag{9}$$

$$U^m(I-Q)\mathfrak{L}_0 \perp (I-Q)\mathfrak{L}_0, \quad U^m(I-Q')\mathfrak{L}_0^* \perp (I-Q')\mathfrak{L}_0^*. \tag{10}$$

for $m = 1, 2, \dots$

In order to prove (9) choose arbitrary vectors $\varphi = Q\varphi'$ and $\psi = Q\psi'$ ($\varphi', \psi' \in \mathfrak{L}_0$). We have

$$(U^m\varphi, \psi) = (U^mQ\varphi', Q\psi') = (QU^mQ\varphi', \psi').$$

Now, $Q\varphi'$ is an element of $\bigvee_{n=1}^{\infty} \mathfrak{L}_n$, hence $QU^mQ\varphi'$ is an element of $\bigvee_{n=m+1}^{\infty} \mathfrak{L}_n$. Thus, by Lemma 1, the last inner product equals 0. This implies the first part of (9), and we can prove its second part in a similar way.

Now, every vector in $(I-Q)\mathfrak{L}_0$ has the form $\varphi=\varphi'-\varphi''$, where $\varphi'\in\mathfrak{L}_0$ and $\varphi''\in\bigvee_{n=1}^{\infty}\mathfrak{L}_n$, thus we have $U^m\varphi\in\bigvee_{n=1}^{\infty}\mathfrak{L}_n$. This implies $U^m\varphi\perp(I-Q)\mathfrak{L}_0$. So we get the first part of (10); the second part can be proved similarly.

Lemma 3. *If* $\mathfrak{W}=\mathfrak{W}^*=\{0\}$, *then* T *is a unitary operator and* $\varrho=1$.

Proof. Let $\mathfrak{W}=\{0\}$. In this case (7) implies $\varphi=Q\varphi$ for every $\varphi=\mathfrak{L}_0$, consequently $\mathfrak{L}_0\subset\bigvee_{n=1}^{\infty}\mathfrak{L}_n$. Now we have $\mathfrak{L}_1=U\mathfrak{L}_0\subset U\bigvee_{n=1}^{\infty}\mathfrak{L}_n=\bigvee_{n=2}^{\infty}\mathfrak{L}_n$, consequently $\mathfrak{L}_0\subset\bigvee_{n=2}^{\infty}\mathfrak{L}_n$ holds too. On the other hand, Lemma 1 shows that $\mathfrak{L}_0\perp\bigvee_{n=2}^{\infty}\mathfrak{L}_n$. So we get $\mathfrak{L}_0\equiv\{0\}$. Hence, $U\varphi=T\varphi$ for $\varphi\in\mathfrak{H}$. We can similarly prove that $\mathfrak{W}^*=\{0\}$ implies $U^*\varphi=T^*\varphi$ for $\varphi\in\mathfrak{H}$. Moreover, $U\varphi=T\varphi$ implies $\|\varphi\|=\|U\varphi\|=\|T\varphi\|=\|\varrho PU\varphi\|=\varrho\|\varphi\|$ for every $\varphi\in\mathfrak{H}$, consequently $\varrho=1$ and we have finished the proof.

Theorem 1. *If* T *is non-unitary, or if* $\varrho\neq 1$, *then* $\sigma(U)$ *is the whole unit circle of the complex plane.*

Proof. Since U is unitary, $\sigma(U)$ is situated on the unite circle. On the other hand, by Lemma 3, there exists an element $\varphi\neq 0$ in $\mathfrak{W}$ or $\mathfrak{W}^*$. By Lemma 2, U is a "bilateral shift" on $\bigvee_{n=-\infty}^{\infty}U^n\varphi$. Since the spectrum of the bilateral shift coincides with the unit circle C so we have a fortiori $\sigma(U)=C$.

A direct proof of the last statement can be given as follows: Suppose the converse case, i. e. that there exists ε such that $|\varepsilon|=1$ and $\varepsilon\notin\sigma(U)$. In this case $(I-\varepsilon U)^{-1}$ is bounded, and using the notation $S_n=I+\varepsilon U+\ldots+(\varepsilon U)^n$, we have $S_n(I-\varepsilon U)=I-(\varepsilon U)^{n+1}$. Hence

$$\|S_n\|=\|[I-(\varepsilon U)^{n+1}](I-\varepsilon U)^{-1}\|\leqq 2\|(I-\varepsilon U)^{-1}\|.$$

Thus $\|S_n\|\leqq K$ with K independent of n. Now choosing φ as above,

$$\|S_n\varphi\|^2=\left\|\sum_{k=0}^{n}(\varepsilon U)^k\varphi\right\|^2=\sum_{k=0}^{n}\|(\varepsilon U)^k\varphi\|^2=(n+1)\|\varphi\|^2,$$

and this contradicts $\|S_n\|\leqq K$.

Lemma 4. *If* T *is completely non-unitary then* $\mathfrak{H}$ *is a subspace of* $\mathfrak{L}\vee\mathfrak{L}^*$.

Proof. If an element φ of $\mathfrak{H}$ can be written as $\varphi=(I-T^{*n}T^n)\psi$ for some n, then we have:

$$\varphi=U^{-1}(U-T)\psi+U^{-2}(U-T)T\psi+\ldots+U^{-n}(U-T)T^{n-1}\psi+$$
$$+U^{-n+1}(U^*-T^*)T^n\psi+U^{-n+2}(U^*-T^*)T^*T^n\psi+\ldots+(U^*-T^*)T^{*n-1}T^n\psi,$$

and, by (5) and (4) this means that $\varphi\in\mathfrak{L}\vee\mathfrak{L}^*$. In case $\varphi=(I-T^nT^{*n})\psi$ for some n, we get the same result by changing the roles of T and T^*, U and U^*.

$\overline{\mathfrak{L}\vee\mathfrak{L}^*}$ is closed, consequently it contains the space spanned by the ranges of $(I-T^{*n}T^n)$ and $(I-T^nT^{*n})$ for all positive integer n. Thus we only have to prove that if, for some $\varphi\in\mathfrak{H}$,

(11) $$\varphi\perp(I-T^{*n}T^n)\mathfrak{H} \quad\text{and}\quad \varphi\perp(I-T^nT^{*n})\mathfrak{H} \qquad (n=1,2,\ldots)$$

then $\varphi=0$. (11) implies

$$(I-T^{*n}T^n)\varphi=0 \quad\text{and}\quad (I-T^nT^{*n})\varphi=0.$$

So we have $T^{*n}T^n\varphi=T^nT^{*n}\varphi=\varphi$, hence $\|T^n\varphi\|^2=\|T^{*n}\varphi\|^2=\|\varphi\|^2$ for $n=1,2,\ldots$. This implies that $\varphi=0$, because T is completely non-unitary.

Lemma 5. *If U is the minimal unitary ϱ-dilation of a completely non-unitary operator T, then $\mathfrak{K}=\mathfrak{L}\vee\mathfrak{L}^*$.*

Proof. Using (2), it suffices to prove that

(12) $$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}=\mathfrak{L}\vee\mathfrak{L}^*.$$

By Lemma 4, $\mathfrak{H}$ is a subspace of $\mathfrak{L}\vee\mathfrak{L}^*$. (4) and (5) imply, that both $\mathfrak{L}$ and $\mathfrak{L}^*$ reduce U, consequently $U^n\mathfrak{H}$ is a subspace of $\mathfrak{L}\vee\mathfrak{L}^*$ for $n=0,\ \pm1,\ldots$. So we have

(12') $$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}\subset\mathfrak{L}\vee\mathfrak{L}^*.$$

On the other hand, (4) implies that both $\mathfrak{L}_k$ and $\mathfrak{L}_k^*$ are contained in $\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}$, for $k=0,\ \pm1,\ldots$. Now (5) shows that

$$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}\supset\mathfrak{L}\vee\mathfrak{L}^*.$$

This relation and (12') prove (12).

Lemma 6. $\mathfrak{L}\vee\mathfrak{L}^*=\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{B}\vee\mathfrak{W}\vee\mathfrak{B}^*\vee\mathfrak{W}^*)$.

Proof. (6) and (7) show that

$$\mathfrak{B}\vee\mathfrak{W}\supset\mathfrak{L}_0 \quad\text{and}\quad \mathfrak{B}^*\vee\mathfrak{W}^*\supset\mathfrak{L}_0^*.$$

So, by (5) and (4)

(13) $$\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{B}\vee\mathfrak{W}\vee\mathfrak{B}^*\vee\mathfrak{W}^*)\supset\mathfrak{L}\vee\mathfrak{L}^*.$$

On the other hand, (4), (5), (6) and (7) show that $\mathfrak{B}$ and $\mathfrak{W}$ are contained in $\mathfrak{L}$; similarly $\mathfrak{B}^*$ and $\mathfrak{W}^*$ are contained in $\mathfrak{L}^*$. Since both $\mathfrak{L}$ and $\mathfrak{L}^*$ reduce U, this implies

$$\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{B}\vee\mathfrak{W}\vee\mathfrak{B}^*\vee\mathfrak{W}^*)\subset\mathfrak{L}\vee\mathfrak{L}^*.$$

This relation and its converse (13) prove the lemma.

Now, by Lemma 3 of [3], the following is true: If U is a unitary operator on

$\mathfrak{K}$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_N$ are wandering subspaces of U, and the set of the finite linear combinations $\sum_{n,k}' \varphi_{n,k}$ $(\varphi_{n,k} \in U^n \mathfrak{A}_k)$ is dense in $\mathfrak{K}$, then U has absolutely continuous spectral measure.

Thus, Lemma 6 implies

Lemma 7. *The restriction of U to the reducing subspace $\mathfrak{L} \vee \mathfrak{L}^*$ has absolutely continuous spectral measure.*

Combining this fact with Lemma 5 we get

Theorem 2. *If U is the minimal unitary ϱ-dilation of a completely non-unitary operator T, then the spectral measure of U is absolutely continuous.*

We shall use the following obvious

Lemma 8. *If T has some unitary ϱ-dilation U with absolutely continuous spectral measure, then T^n converges weakly to O as $n \to \infty$.*

Indeed, using (3) and the Riemann—Lebesgue lemma, we get for $\varphi, \psi \in \mathfrak{H}$

$$(T^n \varphi, \psi) = \varrho(U^n \varphi, \psi) = \varrho \int_0^{2\pi} e^{in\vartheta} \, d(E_\vartheta \varphi, \psi) = \varrho \int_0^{2\pi} e^{in\vartheta} f_{\varphi,\psi}(\vartheta) \, d\vartheta \to 0.$$

Thus Theorem 2 has the following

Corollary. *If T is completely non-unitary and has some unitary ϱ-dilation, then T^n converges weakly to O as $n \to \infty$.*

The next theorem gives a decomposition for T.

Theorem 3. *If T has some unitary ϱ-dilation U, then $\mathfrak{H}$ can be decomposed as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, in such a way that:*

(i) *$\mathfrak{H}_1$ and $\mathfrak{H}_2$ reduce T,*

(ii) *$T_1 = T | \mathfrak{H}_1$ has a unitary ϱ-dilation with absolutely continuous spectral measure,*

(iii) *$T_2 = T | \mathfrak{H}_2$ is unitary.*

Proof. Set $\mathfrak{H}_1 = \mathfrak{H} \cap (\mathfrak{L} \vee \mathfrak{L}^*)$. If $\varphi \in \mathfrak{H}_1$, then

$$T\varphi = (T - U)\varphi + U\varphi \in \mathfrak{L}_0 \vee U(\mathfrak{L} \vee \mathfrak{L}^*) \subset \mathfrak{L} \vee \mathfrak{L}^*$$

thus $T\mathfrak{H}_1 \subset \mathfrak{H}_1$. Similarly, $T^*\mathfrak{H}_1 \subset \mathfrak{H}_1$, so $\mathfrak{H}_1$ reduces T.

Since $\mathfrak{L} \vee \mathfrak{L}^*$ reduces U, the part U_1 of U in $\mathfrak{L} \vee \mathfrak{L}^*$ will be a unitary ϱ-dilation of $T_1 = T | \mathfrak{H}_1$. Now, by Lemma 7, $U_1 = U | (\mathfrak{L} \vee \mathfrak{L}^*)$ has absolutely continuous spectral measure.

It remains to show that if $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, then $T_2 = T | \mathfrak{H}_2$ is unitary. Now, the relations

$$(I - T^*T)\varphi = U^{-1}(U - T)\varphi + (U^* - T^*)T\varphi,$$

$$(I - TT^*)\varphi = U(U^* - T^*)\varphi + (U - T)T^*\varphi$$

$(\varphi \in \mathfrak{H})$ show that $\mathfrak{H}_1$ contains the ranges of both $I-T^*T$ and $I-TT^*$. Thus $\psi \in \mathfrak{H}_2$ implies $\psi \perp (I-T^*T)\mathfrak{H}$ and $\psi \perp (I-TT^*)\mathfrak{H}$, hence $T^*T\psi=\psi$ and $TT^*\psi=\psi$. This means that T is unitary on $\mathfrak{H}_2$, and so we have finished the proof.

References

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