Remarks to a paper of D. Gaier on gap theorems

By GÁBOR HALÁSZ in Budapest

In his paper [1] cited, D. GAIER proved several gap theorems, including the high indices theorem for Borel summability. However, in its original form, his method is not applicable to the theory of Abel summation. In this paper we show how to eliminate this difficulty by a slight modification. At the same time we complete the series of the theorems of GAIER with some more, obtainable by the same modification.

Theorem 1. (HARDY—LITTLEWOOD [2].) *If a series is Abel summable to* 0 *and has Hadamard gaps, then it converges to its Abel sum* 0. *I. e., if*

$$f(x)=\sum_{n=1}^{\infty}a_n e^{-\lambda_n x},\quad \frac{\lambda_{n+1}}{\lambda_n}\geqq q>1,\quad \lambda_1>0$$

and

$$\lim_{x\to+0}f(x)=0,$$

then

$$\sum_{n=1}^{\infty}a_n=0.$$

Or in an almost equivalent form:

Theorem 1'. *If $f(x)$ of Theorem* 1 *is bounded for $x>0$, then*

$$\sup_{N\geqq 1}\left|\sum_{n=1}^{N}a_n\right|\leqq c_1\cdot\sup_{x>0}|f(x)|,$$

where the constant c_1 depends only on the sequence $\{\lambda_n\}$.

(Such positive constants, independent of the quantities a_n, x, N etc. will be denoted in the sequel by $c_2, c_3, \dots$.)

Without any special difficulty, we get the following more precise information about the rapidity of convergence in Theorem 1:

Theorem 2. *Let $r(x)$ $(x>0)$ be an increasing positive function such that with some $\alpha\geqq 0$ $r(x)x^{-\alpha}$ is decreasing. $f(x)$ should satisfy, in addition to the hypotheses of Theorem* 1,

$$|f(x)|\leqq r(x)\qquad (x>0).$$

In this case

$$\Big|\sum_{\lambda_n<X}a_n\Big|\leqq c_2 r(X^{-1}).$$

The condition imposed on $r(x)$ implies that it is larger than constant times x^α near 0. As to smaller remainder terms, let us restrict ourselves to extremely small ones in the following connection.

Theorem 2 suggests that $\sum_{\lambda_n < X} a_n$ and $f(X^{-1})$ have approximately the same order of magnitude and one would expect that an estimation $|f(x)| \leqq \exp\left(-\frac{\text{const.}}{x}\right)$ implies $|\sum_{\lambda_n < X} a_n| \leqq \exp(-\text{const.}\, X)$ so that $f(z)$ is regular in a larger half plane and by its vanishing at $z=0$ of infinite order, $f(z) \equiv 0$. We can deduce this from a weaker estimation, e.g. from $|f(x)| \leqq \exp\left(-\frac{1}{x\log^2 x}\right)$ as is shown by

Theorem 3. *If for the error term we have $r(x) = e^{-s(x)}$ where $s(x)$ is convex from below and*

$$\int_0^1 \sqrt{-s'(x)}\, dx = +\infty$$

then $f(z) \equiv 0$, $a_n \equiv 0$ even if the weaker gap condition

$$\sum_{n=1}^\infty \frac{1}{\sqrt{\lambda_n}} < +\infty \qquad (\lambda_n - \lambda_{n-1} < d)$$

is assumed.

Concerning absolute summability, we prove

Theorem 4. (Zygmund [3].) *If $f(x)$ has Hadamard gaps and is of bounded variation on $(0, +\infty)$, i. e.*

$$\int_0^\infty |f'(x)|\, dx < +\infty,$$

then

$$\sum_{n=1}^\infty |a_n| < +\infty.$$

Theorem 5. *Theorem 4 is valid if $f(x)$, instead of being of bounded variation satisfies the condition*

$$\int_0^\infty \frac{|f(x)|}{x}\, dx < +\infty.$$

Proof of Theorem 1′. Suppose first

$$\sum_{n=1}^\infty \frac{|a_n|}{\lambda_n} < +\infty$$

and consider the Laplace transform

$$F(s) = \int_0^\infty f(x) e^{xs}\, dx \qquad (\operatorname{Re} s < 0).$$

Substituting the series representation of $f(x)$, we get $F(s)$ in the form

$$F(s) = \int_0^\infty \sum_{n=1}^\infty a_n e^{-\lambda_n x + sx}\, dx = \sum_{n=1}^\infty a_n \int_0^\infty e^{(s-\lambda_n)x}\, dx = -\sum_{n=1}^\infty \frac{a_n}{s-\lambda_n},$$

where the change of order of summation and integration is justified by $\sum \int |\ | < +\infty$, a consequence of $\sum_{n=1}^\infty \frac{|a_n|}{\lambda_n} < +\infty$. The sum on the right represents a continuation of $F(s)$ into the right half plane with poles at $s=\lambda_n$ and corresponding residues $-a_n$. An application of the residue theorem gives therefore

$$\sum_{n=1}^N a_n = -\frac{1}{2\pi i} \int_{|s|=R} F(s)\, ds,$$

provided $\lambda_N < R < \lambda_{N+1}$, and our task is to estimate $F(s)$.

On the negative real axis a bound is provided by the original integral representation:

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)| e^{-\sigma x}\, dx \leqq \sup_{x>0} |f(x)| \int_0^\infty e^{-\sigma x}\, dx = \sup_{x>0} |f(x)| \frac{1}{\sigma} \quad (\sigma>0).$$

To be able to extend this, we form the Blaschke product for the region D obtained from the plane by omitting the negative real axis:

$$G(s) = \prod_{n=1}^\infty \frac{1-\dfrac{\sqrt{s}}{\sqrt{\lambda_n}}}{1+\dfrac{\sqrt{s}}{\sqrt{\lambda_n}}} \qquad (\sqrt{s} \text{ is determined by } \sqrt{1}=1).$$

By its vanishing at the poles of $F(s)$, the function

$$H(s) = F(s)G(s)s$$

is regular in the whole of D, satisfying on both sides of the boundary line the inequality

$$|H(-\sigma)| = |F(-\sigma)|\sigma \leqq \sup_{x>0} |f(x)|$$

because $|G(-\sigma)|=1$. Well-known theorems of Lindelöf type (see e. g. [4]) state that the same bound is valid for $H(s)$ in the whole region D if we know in advance some mild estimation on a sequence of circles around 0, tending to infinity. Circles at a distance of at least 1 from all the λ_n's can serve for such a sequence, for we have on them

$$|H(s)| \leqq |F(s)||s| \leqq |s| \sum_{n=1}^{\infty} \frac{|a_n|}{|s-\lambda_n|} \leqq$$

$$\leqq |s| \left\{ \sum_{\lambda_n < \frac{|s|}{2}} \frac{|a_n|}{\lambda_n} + \sum_{\frac{|s|}{2} \leqq \lambda_n < 2|s|} |a_n| + 2 \sum_{2|s| \leqq \lambda_n} \frac{|a_n|}{\lambda_n} \right\} \leqq$$

$$\leqq |s| \left\{ 2 \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} + 2|s| \sum_{\frac{|s|}{2} \leqq \lambda_n < 2|s|} \frac{|a_n|}{\lambda_n} \right\} \leqq 2|s|(|s|+1) \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n}$$

and $O(|s|^2)$ is well below the limit allowed in those theorems of Lindelöf type. Therefore, we may write

$$|H(s)| \leqq \sup_{x>0} |f(x)| \qquad (s \in D).$$

Let $R = \frac{\lambda_{N+1}}{\sqrt{q}}$ in the residue theorem written down above. We shall prove in a Lemma *), using heavily the gap condition (with its notations $\nu_n = \sqrt{\lambda_n}$, $p = \sqrt{q}$, $z = \sqrt{s}$, $m = \sqrt[4]{q}$), that on $|s| = R$ we have $|G(s)| > c_3$, thus by the definition of $H(s)$

$$|F(s)| = \frac{|H(s)|}{|G(s)||s|} \leqq \frac{1}{c_3 R} \sup_{x>0} |f(x)|.$$

The residue theorem then gives

$$\left| \sum_{n=1}^{N} a_n \right| \leqq \frac{1}{2\pi} \int_{|s|=R} |F(s)| \, |ds| \leqq \frac{1}{c_3} \sup_{x>0} |f(x)|.$$

To get rid of the supposition $\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} < +\infty$ we first consider the function $f(x+\delta)$ $(\delta > 0)$ with the coefficients $a_n e^{-\lambda_n \delta}$. For this the above argument holds and hence

$$\left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n \delta} \right| \leqq \frac{1}{c_3} \sup_{x>0} |f(x+\delta)| \leqq \frac{1}{c_3} \sup_{x>0} |f(x)|,$$

and we may let $\delta \to 0$, the bound being uniform in δ, thus completing the proof of Theorem 1'.

P r o o f o f T h e o r e m 1. From Theorem 1' we know already that the partial sums, hence also the coefficients, are bounded so that $\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} < +\infty$ and we may repeat the argument of the previous proof. The only difference is that $f(x)$ is not

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*) See after the proof of Theorem 3.

only bounded but tends to 0 with x, which implies for $\sigma \to +\infty$ the better estimations

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)| e^{-\sigma x}\, dx = \int_0^\infty o(1) e^{-\sigma x}\, dx = o\left(\frac{1}{\sigma}\right),$$

$$|H(-\sigma)| = |F(-\sigma)|\sigma = o(1).$$

Another variant of the Lindelöf type theorem used states that in this case

$$H(s) = o(1) \quad \text{as} \quad |s| \to +\infty,$$

uniformly in s in the whole region D. Returning to $F(s)$, this means

$$F(s) = o\left(\frac{1}{R}\right) \quad \text{for} \quad |s| = R$$

and by the residue theorem

$$\sum_{n=1}^{N} a_n = o(1). \qquad \text{Q. e. d.}$$

Proof of Theorem 2. First we remark that the constant 1 in $r\left(\frac{1}{X}\right)$ has no special significance since if $0 < c < 1$

$$r(cx) \leqq r(x) = x^\alpha r(x) x^{-\alpha} \leqq x^\alpha r(cx)(cx)^{-\alpha} = c^{-\alpha} r(cx),$$

$r(x)$ being increasing, $r(x)x^{-\alpha}$ decreasing.

Now, the proof will consist of a repetition of that of Theorem 1, $o(1)$ replaced everywhere by explicit estimations.

First,

$$|F(-\sigma)| = \int_0^\infty r(x) e^{-\sigma x}\, dx = \int_0^{1/\sigma} + \int_{1/\sigma}^\infty \leqq r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma} + r\left(\frac{1}{\sigma}\right)\sigma^\alpha \int_{1/\sigma}^\infty x^\alpha e^{-\sigma x}\, dx =$$

$$= r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma}\left(1 + \int_1^\infty x^\alpha e^{-x}\, dx\right) = c_4 r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma},$$

i. e.

$$|H(-\sigma)| \leqq c_4 r\left(\frac{1}{\sigma}\right).$$

Owing to the properties of $r(x)$ we can write further with a temporarily fixed R

$$|H(-\sigma)| = c_4 r\left(\frac{1}{R}\right) \quad \text{for} \quad \sigma \geqq R,$$

$$|H(-\sigma)| \leqq c_4 r\left(\frac{1}{\sigma}\right)\sigma^\alpha \frac{1}{\sigma^\alpha} \leqq c_4 r\left(\frac{1}{R}\right) R^\alpha \frac{1}{\sigma^\alpha} = c_4 r\left(\frac{1}{R}\right)\left(\frac{R}{\sigma}\right)^\alpha \quad \text{for} \quad \sigma \leqq R.$$

The test function

$$W(s) = \left(1 + \frac{\sqrt{R}}{\sqrt{s}}\right)^{2\alpha}$$

is regular in D and since $\operatorname{Re}\frac{1}{\sqrt{s}} \geqq 0$ there, we have on both sides of the boundary

$$|W(-\sigma)| \geqq 1 \quad \text{for} \quad \sigma \geqq R,$$

$$|W(-\sigma)| \geqq \left(\frac{\sqrt{R}}{\sqrt{\sigma}}\right)^{2\alpha} = \left(\frac{R}{\sigma}\right)^{\alpha} \quad \text{for} \quad \sigma \leqq R.$$

On account of the estimations of $H(-\sigma)$ from above and those of $W(-\sigma)$ from below, the quotient

$$\frac{H(s)}{W(s)}$$

(which is bounded since $H(s)$ is bounded and $|W(s)| \geqq 1$ in D) has the uniform bound on the negative axis

$$c_4 r\left(\frac{1}{R}\right).$$

By the much used Lindelöf theorem this provides a bound in the whole region and applying it to $|s|=R$

$$|H(s)| \leqq c_4 r\left(\frac{1}{R}\right)|W(s)| = c_4 r\left(\frac{1}{R}\right)\left|1+\frac{\sqrt{R}}{\sqrt{s}}\right|^{2\alpha} \leqq c_4 r\left(\frac{1}{R}\right)2^{2\alpha} = c_5 r\left(\frac{1}{R}\right).$$

For $F(s)$, $|s|=R$ this implies

$$|F(s)| \leqq c_6 r\left(\frac{1}{R}\right)\frac{1}{R},$$

and by the residue theorem

$$\left|\sum_{n=1}^{N} a_n\right| \leqq c_6 r\left(\frac{1}{R}\right) = c_6 r\left(\frac{\sqrt{q}}{\lambda_{N+1}}\right) \leqq c_7 r\left(\frac{1}{\lambda_{N+1}}\right)$$

which is only an alternative formulation of the theorem.

Proof of Theorem 3. The gap condition enables us (and it is its only use here) to form again the product

$$G(s) = \prod_{n=1}^{\infty}\left(1-\frac{\sqrt{s}}{\sqrt{\lambda_n}}\right)\Big/\left(1+\frac{\sqrt{s}}{\sqrt{\lambda_n}}\right).$$

Let further

$$F_\delta(s) = \int_0^\infty f(x+\delta)e^{sx}\,dx, \quad H_\delta(s) = F_\delta(s)G(s)s.$$

As proved in Theorem 1′, the last function is regular and bounded (uniformly with respect to δ) in D with boundary values

$$H_\delta(-\sigma) = -\sigma G(-\sigma)\int_0^\infty f(x+\delta)e^{-\sigma x}\,dx.$$

Since $f(x)\to 0$ as $x\to +0$ and $O(e^{-\mathrm{const}\,x})$ as $x\to+\infty$, these boundary values tend uniformly to

$$H(-\sigma)\overset{\mathrm{def}}{=}-\sigma G(-\sigma)\int_0^\infty f(x)e^{-\sigma x}\,dx\overset{\mathrm{def}}{=}-\sigma G(-\sigma)F(-\sigma)$$

as $\delta\to 0$. But then the convergence is uniform inside D and we get that $H_\delta(s)$ tends to a regular and bounded function $H(s)$ with the above boundary values $H(-\sigma)$. To prove $f(x)\equiv 0$ it is enough to show that $H(s)\equiv 0$. According to a well-known theorem*) this will follow if we can show that

$$\left(\int^\infty+\int^\infty\right)\frac{\log|H(-\sigma)|}{\sigma^{3/2}}\,d\sigma=-\infty,$$

where the integration is on both sides of the boundary line. Recalling $|G(-\sigma)|=1$ and the definition of $H(-\sigma)$, we have to prove this with $F(-\sigma)$ in place of $H(-\sigma)$. Now,

$$|F(-\sigma)|\leqq\int_0^1|f(x)|e^{-\sigma x}\,dx+\int_1^\infty|f(x)|e^{-\sigma x}\,dx\leqq\int_0^1 e^{-s(x)-\sigma x}+O(e^{-\sigma}).$$

For σ large enough the integrand attains its maximum when

$$-s'(x)-\sigma=0.$$

$-s'(x)$ being decreasing, this value $x=x(\sigma)$ is a well-defined and decreasing function of σ and with it the integral is less than

$$1\cdot e^{-s(x)-\sigma x}\leqq e^{-\sigma x}.$$

Since $x(\sigma)$ tends to 0 with $1/\sigma$, the second term in the estimation of $F(-\sigma)$ cannot exceed this bound and hence

$$|F(-\sigma)|=O(e^{-\sigma x})\qquad\log|F(-\sigma)|\leqq-\sigma x+O(1),$$

and it suffices to show

$$\int^\infty\frac{\sigma x}{\sigma^{3/2}}\,d\sigma=\int^\infty\frac{x}{\sqrt{\sigma}}\,d\sigma=+\infty.$$

Introducing x as a new variable, we have, since $-s'(x)\equiv\sigma$,

$$\int^\infty\frac{x}{\sqrt{\sigma}}\,d\sigma=-\int_0\frac{x}{\sqrt{\sigma}}\sigma'(x)\,dx=\int_0\frac{xs''(x)}{\sqrt{-s'(x)}}\,dx$$

and partial integration shows that this is

$$\geqq O(1)+\varliminf_{x=0}x2\sqrt{-s'(x)}+2\int_0\sqrt{-s'(x)}\,dx\geqq O(1)+2\int_0\sqrt{-s'(x)}\,dx=+\infty.$$

Qu. e. d.

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*) This is JENSEN's inequality $\int_{|z|=1}\log|f(z)||d_z|\geqq\log|f(0)|$, after a conformal mapping of the unit disk onto our region D.

Lemma. *If* $\{v_n\}$ *is an Hadamard sequence* $\left(\frac{v_{n+1}}{v_n} \geqq p > 1\right)$ *of positive numbers then with the definition*

$$B(z) = \prod \left(1-\frac{z}{v_n}\right) \Big/ \left(1+\frac{z}{v_n}\right)$$

we have

$$c_8 < |B(z)| < c_9$$

for $v_N m < |z| < \frac{v_{N+1}}{m}$ $(m > 1)$, *uniformly in* z *and* N.

Proof. For $n \geqq N+1$

$$\left|\frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \frac{1+\frac{|z|}{v_n}}{1-\frac{|z|}{v_n}} < e^{c_{10}\frac{|z|}{v_n}},$$

$$\left|\prod_{n \geqq N+1} \frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| < \exp\left[c_{10}|z| \sum_{n \geqq N+1} \frac{1}{v_n}\right] \leqq \exp\left[c_{10}\frac{|z|}{v_{N+1}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)\right] \leqq$$

$$\leqq \exp\left[c_{11}\frac{|z|}{v_{N+1}}\right] < e^{c_{11}}.$$

For $n \leqq N$

$$\left|\frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \frac{\frac{|z|}{v_n}+1}{\frac{|z|}{v_n}-1} = 1+\frac{2}{\frac{|z|}{v_n}-1} \leqq 1+\frac{2}{\left(1-\frac{1}{m}\right)\frac{|z|}{v_n}} \leqq e^{c_{12}\frac{v_n}{|z|}},$$

$$\left|\prod_{n \leqq N} \frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \exp\left(c_{12}\frac{1}{|z|}\sum_{n \leqq N} v_n\right) \leqq \exp\left[c_{12}\frac{v_N}{|z|}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)\right] \leqq$$

$$\leqq \exp\left(c_{13}\frac{v_N}{|z|}\right) < e^{c_{13}},$$

thus $B(z)$ is bounded above. Since $B(z) = \frac{1}{B(-z)}$, boundedness from below follows.

Proof of Theorem 4. The hypotheses of Theorem 1′ are satisfied here and we can use the results of its proof.

Separating real and imaginary parts, we may suppose a_n real. If and only if

a_n and a_{n+1} are of different sign we pick out a number between λ_n and λ_{n+1}, e. g. their geometric mean and denote the sequence constructed this way by $\{\mu_k\}$. On the positive real axis the function

$$P(s) = \prod_{k=1}^{\infty}\left(1-\frac{s}{\mu_k}\right)\Big/\left(1+\frac{s}{\mu_k}\right)$$

changes sign at its simple zeros $s=\mu_k$ so that according to the construction, $P(\lambda_n)$ has the same changes of sign as a_n, and $a_n P(\lambda_n)$ is either always positive or always negative. Now the μ_k's form an Hadamard sequence and since λ_n is "far" from all of them in the sense required in the lemma, applying it to $P(s)$,

$$|P(\lambda_n)| > c_{14}.$$

Consequently

$$\sum_{n=1}^{N} |a_n| \leqq \frac{1}{c_{14}} \sum_{n=1}^{N} |a_n P(\lambda_n)| = \frac{1}{c_{14}} \left| \sum_{n=1}^{N} a_n P(\lambda_n) \right|,$$

and it is enough to prove that the sum on the right is bounded.

This is a partial sum of the residues of the function $F(s)P(s)$ originating from the poles of $F(s)$ at $s=\lambda_n$. Taking into account the remaining poles $s=-\mu_k$ provided by $P(s)$:

$$\sum_{\lambda_n<R} a_n P(\lambda_n) = -\frac{1}{2\pi i} \int_{|s|=R} F(s)P(s)\,ds + \sum_{\mu_k<R} F(-\mu_k) \operatorname*{Res}_{s=-\mu_k} P(s).$$

Let here $R = \dfrac{\lambda_N}{\sqrt[4]{q}}$. This choice guarantees that R is far from both the λ_n's and the μ_k's and we have

$$F(s) = O\left(\frac{1}{R}\right) \qquad (|s| = R)$$

as we proved in Theorem 1′ while

$$|P(s)| < c_{15} \ (|s| = R)$$

by the Lemma so that the integral on the right hand side is bounded.

To estimate the sum on the right, we return to the original integral representation of $F(s)$:

$$F(-\mu_k) = \int_0^{\infty} f(x) e^{-\mu_k x}\,dx,$$

$$\sum_{\mu_k<R} F(-\mu_k) p_k = \int_0^{\infty} f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx.$$

Here we have introduced the notation $p_k = \operatorname*{Res}_{s=-\mu_k} P(s)$. Integrating by parts where we may assume that $\lim_{x=0} f(x)=0$ making the integrated part vanish,

$$\left|\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx\right| = \left|\int_0^\infty f'(x) \sum_{\mu_k<R} \frac{p_k}{\mu_k} e^{-\mu_k x}\,dx\right| \leqq$$

$$\leqq \max_{x\geqq 0}\left|\sum_{\mu_k<R}\frac{p_k}{\mu_k}e^{-\mu_k x}\right|\cdot\int_0^\infty |f'(x)|\,dx \leqq \sup_K\left|\sum_{k=1}^K \frac{p_k}{\mu_k}\right|\int_0^\infty |f'(x)|\,dx,$$

the last inequality by a simple Abelian theorem. Appealing once again to the residue theorem, we have

$$\sum_{\mu_k<R}\frac{p_k}{\mu_k} = -\frac{1}{2\pi i}\int_{|s|=R}\frac{P(s)}{s}\,ds+1$$

where the estimation $|P(s)|<c_{15}$ assures that the integral is bounded and so are the partial sums of $\sum_{k=1}^\infty \frac{p_k}{\mu_k}$. Thus we proved

$$\sum_{k=1}^K F(-\mu_k)\operatorname*{Res}_{s=-\mu_k} P(s) = O(1),$$

hence

$$\sum_{n=1}^N a_n P(\lambda_n) = O(1),$$

from which according to an earlier remark the conclusion already follows.

P r o o f o f T h e o r e m 5. We try to repeat the previous proof up to the stage where the boundedness of variation was exploited.

The results of Theorem 1′ were used there and we do not know in advance if its hypotheses are fulfilled here. But a simple consequence of our assumptions is

$$\int_0^\infty |f(x)|\,dx < +\infty$$

and thus the primitive function

$$\int f(x)\,dx = -\sum_{n=1}^\infty \frac{a_u}{\lambda_n}e^{-\lambda_n x}$$

is of bounded variation on $(0, +\infty)$, implying by Theorem 4

$$\sum_{n=1}^\infty \frac{|a_n|}{\lambda_n} < +\infty.$$

This was a prerequisite for the considerations in the proof of Theorem 1′. The boundedness of $f(x)$ was used to deduce $F(-\sigma)=O\left(\frac{1}{\sigma}\right)$. Here we have, however, the same bound by

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)|e^{-\sigma x}\,dx = \int_0^\infty \frac{|f(x)|}{x} xe^{-\sigma x}\,dx \leqq$$

$$\leqq \max_{x\geqq 0} xe^{-\sigma x}\int_0^\infty \frac{|f(x)|}{x}\,dx = \frac{e^{-1}}{\sigma}\int_0^\infty \frac{|f(x)|}{x}\,dx.$$

Therefore the results of the proof of Theorem 1′ are valid,

$$F(s) = O\left(\frac{1}{R}\right) \qquad (|s| = R)$$

and the proof holds unaltered until the partial integration. This was performed to prove the boundedness of

$$\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx.$$

Here we proceed more roughly:

$$\left|\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx\right| \leqq \int_0^\infty |f(x)| \sum_{k=1}^\infty |p_k| e^{-\mu_k x}\,dx.$$

As we showed, the partial sums of $\sum_{k=1}^\infty \frac{p_k}{\mu_k}$ are bounded, hence $|p_k|<c_{16}\mu_k$ and we get further, $\{\mu_k\}$ being an Hadamard sequence,

$$\sum_{k=1}^\infty |p_k|e^{-\mu_k x} \leqq c_{16}\sum_{k=1}^\infty \mu_k e^{-\mu_k x} \leqq c_{17}\sum_{k=1}^\infty (\mu_k-\mu_{k-1})e^{-\mu_k x} \leqq$$

$$\leqq \frac{c_{17}}{x}\sum_{k=1}^\infty \int_{\mu_{k-1}x}^{\mu_k x} e^{-u}\,du \leqq \frac{c_{17}}{x}.$$

The integral to be estimated is, therefore, less than

$$c_{17}\int_0^\infty \frac{|f(x)|}{x}\,dx$$

and with this the proof is completed.

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References

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