The spectrum of the Cesàro operator

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Introduction

Suppose that x is a locally integrable function on $R^+ = [0, \infty)$ and that the Cesàro average of x is defined by

(1) $$Px(t) = \frac{1}{t} \int_0^t x(s)\, ds.$$

In [3], BROWN, HALMOS and SHIELDS considered the operator P as a bounded operator from $L^2(R^+)$ to itself and showed that the spectrum in this case is the circle

(2) $$\sigma(P; L^2) = \{\lambda : |\lambda - 1| = 1\}.$$

In this paper, we examine P as an operator in $L^p(R^+)$ when $p \neq 2$ and show that the spectrum in this case is the following set:

(3) $$\sigma(P; L^p) = \{\lambda : \mathrm{Re}\,(1/\lambda) = (p-1)/p\},$$

which, for $p > 1$, is a circle with centre $2(p-1)/p$ and the same radius, and for $p = 1$, is the imaginary axis.

The result can be extended to include certain rearrangement invariant spaces X, in which case the spectrum becomes the following lune:

(4) $$\sigma(P; X) = \{\lambda : 1 - \beta \leqq \mathrm{Re}\,(1/\lambda) \leqq 1 - \alpha\},$$

where α and β are the indices associated with the space X as in [1]. The proof for this will appear elsewhere.

The method of proof is to exhibit integral operators which are proved to be the resolvents of P for $\mathrm{Re}\,(1/\lambda) < (p-1)/p$ and $\mathrm{Re}\,(1/\lambda) > (p-1)/p$, respectively. A short additional argument then shows that the spectrum is indeed given by (3).

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Preliminary Lemmas

Let x be a locally integrable function, and let ζ be a complex number. Define the operators P_ζ and Q_ζ by

$$P_\zeta x(t) = \int_0^1 s^{-\zeta} x(st)\, ds, \tag{5}$$

whenever

$$\int_0^1 |s^{-\zeta} x(st)|\, ds < \infty \quad \text{a.e.},$$

and

$$Q_\zeta x(t) = \int_1^\infty s^{-\zeta} x(st)\, ds, \tag{6}$$

whenever

$$\int_1^\infty |s^{-\zeta} x(st)|\, ds < \infty \quad \text{a.e.}$$

We denote the space of bounded linear operators on L^p by $B(L^p)$ and the spectral radius and norm of $T \in B(L^p)$ by $r(T; L^p)$ and $\|T\|_p$, respectively.

Lemma 1. *Let* $1 \leqq p \leqq \infty$, *and the operators* P_ζ *and* Q_ζ *be defined by* (5) *and* (6).
(a) $P_\zeta \in B(L^p)$ *with domain all of* L^p *if and only if*

$$\operatorname{Re} \zeta < (p-1)/p \quad (=1, \text{ if } p = \infty). \tag{7}$$

In this case,

$$\|P_\zeta\|_p = r(P_\zeta; L^p) = \left[\frac{p-1}{p} - \operatorname{Re} \zeta\right]^{-1}. \tag{8}$$

(b) $Q_\zeta \in B(L^p)$ *with domain all of* L^p *if and only if*

$$\operatorname{Re} \zeta > (p-1)/p. \tag{9}$$

In this case,

$$\|Q_\zeta\|_p = r(Q_\zeta; L^p) = \left[\operatorname{Re} \zeta - \frac{p-1}{p}\right]^{-1}. \tag{10}$$

Proof. The proof that (7) implies that $P_\zeta \in B(L^p)$ and that (9) implies that $Q_\zeta \in B(L^p)$ can be derived from ([4], Th. 318). The other parts are given for real ζ in ([2], Theorem 2 and introductory remarks), and the proofs given there are easily extended to complex ζ.

Lemma 2. *Let* $1 \leqq p \leqq \infty$. *Let* $x \in L^p$ *be such that* $Px \in L^p$.
(a) *If* $P_\zeta \in B(L^p)$, *then* $PP_\zeta x \in L^p$, *and*

$$\zeta P P_\zeta x = \zeta P_\zeta P x = (P - P_\zeta) x, \tag{11}$$

(b) *If* $Q_\zeta \in B(L^p)$, *then* $PQ_\zeta x \in L^p$, *and*

$$\zeta PQ_\zeta x = \zeta Q_\zeta Px = (P+Q_\zeta)x. \tag{12}$$

Proof. (a) Since $Px \in L^p$, and $P_\zeta \in B(L^p)$, $P_\zeta Px \in L^p$, and

$$\int_0^1 |s^{-\zeta}(Px)(st)|\, ds < \infty, \quad t > 0. \tag{13}$$

We can write (13) as an iterated integral using the definition of P to show that

$$\int_0^1 s^{-\mathrm{Re}\,\zeta-1}\, ds \int_0^s |x(ut)|\, du < \infty, \quad t > 0, \tag{14}$$

and then apply FUBINI's theorem to the following iterated integral

$$\begin{aligned} \zeta P_\zeta Px(t) &= \zeta \int_0^1 s^{-\zeta-1}\, ds \int_0^s x(ut)\, du = \zeta \int_0^1 x(ut)\, du \int_u^1 s^{-\zeta-1}\, ds = \\ &= \int_0^1 (1-u^{-\zeta})x(ut)\, du = Px(t) - P_\zeta x(t). \end{aligned} \tag{15}$$

Also, changing variables in another way and using (14) to justify the interchange of order of integration,

$$P_\zeta Px(t) = \int_0^1 s^{-\zeta}\, ds \int_0^1 x(sut)\, du = \int_0^1 du \int_0^1 s^{-\zeta} x(sut)\, du = PP_\zeta x(t), \quad t > 0. \tag{16}$$

This proves (11). (Note that $\mathrm{Re}\,\zeta < 1$ is necessary for $P_\zeta \in B(L^p)$ by Lemma 1, so we have used this fact freely.)

(b) The proof of (12) follows the same pattern as in (a), and we leave the appropriate manipulations to the reader.

The resolvent of P

By Lemma 1, applied to $\zeta = 0$, it is clear that $P \in B(L^p)$ iff $1 < p \leqq \infty$. Of course, this is a well known result of HARDY. In case $p=1$ we can define P as a closed linear operator with range L^1 and domain $D(P; L^1)$ dense in L^1 by the simple expedient of defining

$$D(P; L^1) = \left\{ x \in L^1 : \int_0^1 dt \int_0^1 |x(st)|\, ds < \infty \right\}. \tag{17}$$

To show $D(P; L^1)$ is dense in L^1, we note that it contains all functions in L^1 vanishing in a neighbourhood of 0. Since convergence in norm in L^1 implies convergence a.e., it is easy to prove that P is closed as an operator $D(P; L^1) \to L^1$.

For $p>1$, we define $D(P; L^p)=L^p$.

The resolvent set of P considered as an operator $D(P; L^p)\to L^p$ will be denoted $\varrho(P; L^p)$ and the spectrum by $\sigma(P; L^p)$.

Theorem 1. *Let λ be a complex number satisfying*

$$\operatorname{Re}(1/\lambda)<(p-1)/p \quad or \quad \operatorname{Re}(1/\lambda)>(p-1)/p. \tag{18}$$

Then, $\lambda\in\varrho(P; L^p)$, and for each $x\in L^p$,

$$(\lambda-P)^{-1}x=(\lambda^{-1}+\lambda^{-2}P_{1/\lambda})x, \quad \operatorname{Re}(1/\lambda)<(p-1)/p, \tag{19}$$

$$(\lambda-P)^{-1}x=(\lambda^{-1}-\lambda^{-2}Q_{1/\lambda})x, \quad \operatorname{Re}(1/\lambda)>(p-1)/p. \tag{20}$$

Proof. Let $\zeta=\lambda^{-1}$. And $\operatorname{Re}(\zeta)<(p-1)/p$. From Lemma 1, we have $P_\zeta\in B(L^p)$, and from Lemma 2,

$$(\lambda-P)(\zeta+\zeta^2P_\zeta)x=[I-\zeta P+\zeta P_\zeta-\zeta^2PP_\zeta]x=x, \tag{21}$$

and also

$$(\zeta+\zeta^2P_\zeta)(\lambda-P)x=x, \tag{22}$$

for every $x\in D(P; L^p)$. But $D(P; L^p)$ is dense in L^p and hence (21) and (22) are enough to show that $(\lambda-P)$ has the bounded inverse $\zeta+\zeta^2P_\zeta$, for $\operatorname{Re}(\zeta)<(p-1)/p$.

Similarly, $(\lambda-P)$ has the bounded inverse given in (20) for $\operatorname{Re}(\zeta)>(p-1)/p$.

Theorem 2. *Let λ be a complex number satisfying* $\operatorname{Re}(1/\lambda)=(p-1)/p$. *Then $\lambda\in\sigma(P; L^p)$.*

Proof. Let λ_n be a sequence of complex numbers with $\operatorname{Re}(1/\lambda_n)<(p-1)/p$, approaching λ. Then by Lemma 1, if $\zeta_n=\lambda_n^{-1}$,

$$\|\zeta_n+\zeta_n^2P_\zeta\|_p\geqq|\zeta_n|^2\|P_{\zeta_n}\|-|\zeta_n|=|\zeta_n|^2\cdot[(p-1)/p-\operatorname{Re}\zeta_n]^{-1}-|\zeta_n|\to\infty$$

as $\zeta_n\to\zeta$. Hence $\lambda\in\sigma(P; L^p)$.

References

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(Received April 12, 1967)