Inequalities and theorems concerning strongly multiplicative systems

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Introduction

ALEXITS introduced the following definitions (see [1], p. 186).

The sequence of real measurable functions $\varphi_1(t), \varphi_2(t), \cdots$ defined in the interval [0, 1], is called a multiplicative system if all their finite products are Lebesgue-integrable with

$$(1) \qquad \int_0^1 \varphi_{n_1}(t)\varphi_{n_2}(t)\cdots\varphi_{n_k}(t)\,dt = 0 \qquad (n_1 < n_2 < \cdots < n_k;\ k=1,2,\cdots).$$

The sequence $\{\varphi_n(t)\}$ is called a strongly multiplicative system (SMS) if the system $\{\varphi_{n_1}(t)\varphi_{n_2}(t)\cdots\varphi_{n_k}(t)\}$ is an orthogonal system, i.e.

$$(2) \qquad \int_0^1 \varphi_{n_1}^{\alpha_1}(t)\varphi_{n_2}^{\alpha_2}(t)\cdots\varphi_{n_k}^{\alpha_k}(t)\,dt = 0 \qquad (n_1 < n_2 < \cdots < n_k;\ k = 1,2,\cdots),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$ can be equal to 1 or 2 but at least one element of the sequence $\alpha_1, \alpha_2, \cdots, \alpha_k$ is equal to 1.

The sequence $\{\varphi_n(t)\}$ is called an equinormed strongly multiplicative system (ESMS) if the system $\{\varphi_{n_1}(t)\varphi_{n_2}(t)\cdots\varphi_{n_k}(t)\}$ is an orthogonal and normal system, i.e.

$$\int_0^1 \varphi_n(t)\,dt = 0, \quad \int_0^1 \varphi_n^2(t)\,dt = 1 \qquad (n=1,2,\cdots);$$

$$(3) \qquad \int_0^1 \varphi_{n_1}^{\alpha_1}(t)\varphi_{n_2}^{\alpha_2}(t)\ldots\varphi_{n_k}^{\alpha_k}(t)dt =$$

$$= \int_0^1 \varphi_{n_1}^{\alpha_1}(t)\,dt \int_0^1 \varphi_{n_2}^{\alpha_2}(t)\,dt \cdots \int_0^1 \varphi_{n_k}^{\alpha_k}(t)\,dt \quad (n_1 < n_2 < \cdots < n_k;\ k=1,2,\ldots),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$ can be equal to 1 or 2.

Evidently a sequence of independent functions (with mean value 0 and dispersion 1) is an ESMS. Another example is a strongly lacunary sequence of trigonometric functions, i.e. $\{\sqrt{2}\sin 2\pi n_k t\}$ if $n_{k+1}/n_k \geqq 3$ $(k=1, 2, \cdots)$.

ALEXITS proved that an ESMS has the property of the independent functions, i. e. $\sum_{n=1}^{\infty} a_n\varphi_n(t)$ is convergent almost everywhere if and only if $\sum_{n=1}^{\infty} a_n^2 < \infty$. More exactly he proved

Theorem A. *If $\{\varphi_n(t)\}$ in a uniformly bounded* ESMS *then under the condition $\sum_{n=1}^{\infty} a_n^2 < \infty$ the series $\sum_{n=1}^{\infty} a_n\varphi_n(t)$ is convergent almost everywhere. Furthermore, if for every measurable set $E\subset[0, 1]$ and for sufficiently large n the relation*

$$\int_E \varphi_n^2(t)\,dt \geqq C\,\mathrm{mes}\,(E) \quad [1]$$

holds (where C is a positive constant depending only on E), and if the series $\sum_{n=1}^{\infty} a_n\varphi_n(t)$ is convergent in a set of positive measure then $\sum_{n=1}^{\infty} a_n^2 < \infty$.

(In [1] this theorem is given in a more general form.)

The aim of the present paper is to study what other properties of the independent functions remain valid for an ESMS. Namely we prove the inequality due to BERNSTEIN and other exponential bounds, furthermore, the central limit theorem and a weaker form of the law of iterated logarithm for ESMS. Let me recall here the well-known forms of these theorems.

We shall use, for any sequence $\{\varphi_n(t)\}$ of functions, the following notations:

$$S_N(t) = \sum_{n=1}^{N} a_n\varphi_n(t), \quad A_N^2 = \sum_{n=1}^{N} a_n^2, \quad M_N = \max_{1\leqq n\leqq N} |a_n| \qquad (N=1, 2, \cdots).$$

The following inequality is due to BERNSTEIN [2]:

Theorem B. *Let $\{\varphi_n(t)\}$ be a system of independent functions on* [0, 1] *with mean value* 0 *and dispersion* 1, *and uniformly bounded by the constant K, furthermore, let x be a positive real number such that*

$$\vartheta = \frac{xM_N K}{A_N^2} \leqq 1.$$

Then

$$\mathrm{mes}\,(\{S_N(t) \geqq x\}) \leqq \exp\left\{-\frac{x^2}{2A_N^2}(1-\vartheta)\right\}.$$

[1] mes (E) denotes the Lebesgue measure of the set E.

Here we prove the following analogous form:

Theorem 1. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS, *with bound K, and let x be a positive real number. Then*

$$\text{mes}\,(\{S_N(t) \geqq x\}) \leqq \exp\left\{-\frac{x^2}{2A_N^2}(1-\theta)\right\} \quad \textit{with} \quad \theta = \frac{xM_N K^3}{A_N^2}. \tag{4}$$

Remark 1. We observe if $S_N(t)$ is replaced by $-S_N(t)$ the conclusion yields

$$\text{mes}\,(\{|S_N(t)| \geqq x\}) \leqq 2\exp\left\{-\frac{x^2}{2A_N^2}(1-\theta)\right\} \quad \text{with} \quad \theta = \frac{xM_N K^3}{A_N^2}.$$

We show that the reverse inequality also holds if xM_N/A_N^2 is sufficiently small and x^2/A_N^2 is sufficiently large, the analogous form of which can be found in the quoted paper of Kolmogoroff [2].

Theorem 2. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS, *with bound K, and let x be a positive real number. If the inequalities*

$$\text{(i)}\ \frac{xM_N K^3}{A_N^2} = \alpha \leqq \frac{1}{2^{13}} \quad \textit{and} \quad \text{(ii)}\ \frac{x^2}{A_N^2} = \beta \geqq 2^{14} \tag{5}$$

are satisfied, then

$$\text{mes}\,(\{S_N(t) \geqq x\}) \geqq \exp\left\{-\frac{x^2}{2A_N^2}(1+\varepsilon)\right\}, \tag{6}$$

where

$$\varepsilon = \max\left\{64\sqrt{2\alpha},\ 32\sqrt{\frac{\log\beta}{\beta}}\right\}.$$

Marcinkiewicz and Zygmund [3] proved the following [2])

Theorem C. *Let $\{\varphi_n(t)\}$ be a system of independent functions on* $[0, 1]$, *with mean value* 0 *and dispersion* 1. *Then, for all positive real numbers p (>1), we have*

$$\bar{C}_p A_N \leqq \left\{\int_0^1 \left(\max_{1\leqq n\leqq N} |S_n(t)|^p\, dt\right)\right\}^{\frac{1}{p}} \leqq \bar{D}_p A_N, \tag{7}$$

where $\bar{C}_p$ and $\bar{D}_p$ are positive constants depending only on p.

An essentially similar result holds for lacunary trigonometric series [3]), too

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[2]) Here we give the original theorem with a little modification.

[3]) $\sum_{k=1}^{\infty}(a_k \cos n_k t + b_k \sin n_k t)$ in called lacunary if $n_{k+1}/n_k \geqq q > 1$ $(k = 1, 2, \ldots)$.

(see [4], v. 1, p. 203). Unfortunately we cannot assert the analogous result for ESMS, but the following result is valid:

Theorem 3. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded normed* SMS. *Then, for all positive real numbers* p, *we have*

$$C_p A_N \leqq \left\{\int_0^1 |S_N(t)|^p \, dt\right\}^{\frac{1}{p}} \leqq D_p A_N, \tag{8}$$

where C_p *and* D_p *are positive constants depending only on* p. *Furthermore, if for* A_n *and a positive real number* λ *we have*

$$A_N \leqq 1 \quad \text{and} \quad \lambda \leqq \frac{1}{8eK^6}, \tag{9}$$

then

$$\int_0^1 \exp\{\lambda S_N^2(t)\} \, dt \leqq 2. \tag{10}$$

Moreover, we succeeded in proving the following theorem (for the case Rademacher functions, see [4], v. II, p. 235.):

Theorem 4. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded normed* SMS. *Then the following estimations are valid:*

$$CA_N \log^+ A_N - C' \leqq \int_0^1 |S_N(t)| \log^+ |S_N(t)| \, dt \leqq CA_N \log^+ A_N + C', \tag{11}$$ [4])

where C *and* C' *are positive absolute constants.*

Remark 2. It will be clear from the proofs that both Theorem 3 and Theorem 4 remain valid if $S_N(t)$ and A_N are replaced by $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ and $A^2 = \sum_{n=1}^{\infty} a_n^2$ in them supposing that $A < \infty$ or $A \leqq 1$, respectively. In particular, if $\sum_{n=1}^{\infty} a_n^2 < \infty$ then the sum of $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ belongs to L^p for every positive real number p.

Concerning the law of iterated logarithm, the basic result, obtained by Kolmogoroff [2], reads as follows:

Theorem D. *Let* $\{\varphi_n(t)\}$ *be a system of bounded independent functions on* $[0, 1]$, *with mean value* 0 *and dispersion* 1. *If*

$$\text{(i)} \; A_N \to \infty, \qquad \text{(ii)} \; |a_N \varphi_N(t)| \leqq m_N = o\left(\sqrt{\frac{A_N^2}{\log\log A_N^2}}\right), \tag{12}$$

[4]) By $\log^+|u|$ we mean $\log|u|$ wherever $|u| \geqq 1$, and 0 otherwise.

then

$$\text{mes}\left(\left\{\limsup_{N\to\infty} \frac{S_N(t)}{\sqrt{2A_N^2 \log\log A_N^2}} = 1\right\}\right) = 1. \tag{13}$$

For lacunary trigonometric series SALEM and ZYGMUND [5] have shown that under the hypotheses (12) we have (13) with "$\leqq$" instead of "$=$". In this case a complete proof of (13) was given later by ERDŐS and GÁL [6]. Recently, RÉVÉSZ [7] obtained the following result:

Theorem E. *If $\{\varphi_n(t)\}$ is a uniformly bounded* ESMS, *then*

$$\text{mes}\left(\left\{\limsup_{N\to\infty} \frac{\varphi_1(t)+\varphi_2(t)+\dots+\varphi_N(t)}{\sqrt{N\log\log N}} \leqq 6\right\}\right) = 1.$$

We managed to prove the following result which can be roughly formulated as follows: if the sequence of indices $m_1 < m_2 < \cdots$ is rare enough, then the law of iterated logarithm will be valid for the subsequence $\{S_{m_k}(t)\}$ with "$\leqq$" instead of "$=$". More exactly, we prove

Theorem 5. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS. *Under the conditions*

$$\text{(i) } A_N \to \infty \quad \textit{and} \quad \text{(ii) } M_N = o\left(\sqrt{\frac{A_N^2}{\log\log A_N^2}}\right), \tag{14}$$

for every positive real number ε there exists a sequence of natural numbers $N_1 < N_2 < \cdots$ having the following property: if $m_1 < m_2 < \cdots$ is an arbitrary sequence of natural numbers for which $N_k \leqq m_k < N_{k+1}$ $(k = 1, 2, \cdots)$, then we have

$$\text{mes}\left(\left\{\limsup_{k\to\infty} \frac{S_{m_k}(t)}{\sqrt{2A_{m_k}^2 \log\log A_{m_k}^2}} \leqq 1+\varepsilon\right\}\right) = 1. \tag{15}$$

Remark 3. It will be clear from the proof that if we had the stronger inequality (7) for a uniformly bounded ESMS too, then under the hypotheses (14) we could assert also (13) with "$\leqq$" instead of "$=$". Unfortunately, we only have the weaker inequality (8) for a uniformly bounded normed SMS.

A number of authors have generalized the central limit theorem for the lacunary trigonometric series. The most general result is due to SALEM and ZYGMUND [8], who state the following

Theorem F. *Let $S_N(t)$ denote the Nth partial sum of the lacunary trigonometric series $\sum_{k=1}^{\infty} (a_k \cos n_k t + b_k \sin n_k t)$, $n_{k+1}/n_k \geqq q > 1$ $(k = 1, 2, \cdots)$, and let $a_1, a_2, \cdots; b_1, b_2, \cdots$ be arbitrary sequences of real numbers for which*

$$C_N = \left\{\frac{1}{2}\sum_{k=1}^{N} (a_k^2 + b_k^2)\right\}^{\frac{1}{2}} \to \infty \quad \text{and} \quad \{a_N^2 + b_N^2\}^{\frac{1}{2}} = o(C_N).$$

Then, for any set $E \subset [0, 2\pi]$ *of positive measure, the distribution functions*

$$F_N(y; E) = \frac{\text{mes}(\{t \in E : S_N(t)/C_N \leqq y\})}{\text{mes}(E)} \qquad (N=1, 2, \cdots)$$

tend to the Gaussian distribution with mean value 0 *and dispersion* 1.

We obtained the following result:

Theorem 6. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS. *If*

(16) $$A_N \to \infty \quad and \quad a_N = o(A_N);$$

then the distribution functions

(17) $$F_N(y) = \text{mes}\left(\left\{\frac{S_N(t)}{A_N} \leqq y\right\}\right) \qquad (N=1, 2, \cdots)$$

tend pointwise to the Gaussian distribution function

$$G(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du.$$

This theorem contains a result of Révész [7] (case $a_n = 1$ for every n).

§ 1. The proof of Theorem 1 and Theorem 2

The following lemma has a fundamental significance in the proof of Theorem 1 and Theorem 2.

Lemma 1. *Let* λ *be an arbitrary non-negative real number. Then*

(18) $$\exp\left\{\frac{\lambda^2 A_N^2}{2}\left(1 - \frac{\lambda^2 M_N^2}{2} - \lambda M_N K^3\right)\right\} \leqq \int_0^1 \exp\{\lambda S_N(t)\}\, dt \leqq \leqq \exp\left\{\frac{\lambda^2 A_N^2}{2}(1 + \lambda M_N K^3)\right\}.$$

Proof. For every real number u, we have that

(19) $$\left|\log\left(1 + u + \frac{u^2}{2}\right) - u\right| \leqq \frac{|u|^3}{2}.$$ [5]

[5]) (19) follows from the sharper estimates: $0 \leqq u - \log(1+u+u^2/2) \leqq u^3/3$ for $u \geqq 0$ and $u^3/3 \leqq \leqq u - \log(1+u+u^2/2) \leqq 0$ for $u \leqq 0$. We only have to remark that the function $\varkappa(u) = = u - \log(1+u+u^2/2)$ is non-decreasing $-\infty < u < \infty$ and $\varkappa(0) = 0$, and that the function $\mu(u) = = u - \log(1+u+u^2/2) - u^3/3$ is non-increasing and $\mu(0) = 0$.

Applying this inequality, we get that

$$\exp\{\lambda a_n \varphi_n(t)\} = \left(1+\lambda a_n \varphi_n(t) + \frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2}\right) \exp\{R_n(t)\},$$

where

$$|R_n(t)| \leqq \frac{\lambda^3 K^3 M_N a_n^2}{2} \qquad (n=1, 2, \ldots, N).$$

Hence

$$\text{(20)} \quad \int_0^1 \exp\{\lambda S_N(t)\}\, dt \leqq \prod_{n=1}^{N} \exp\left\{\frac{\lambda^3 K^3 M_N a_n^2}{2}\right\} \int_0^1 \prod_{n=1}^{N} \left(1+\lambda a_n \varphi_n(t) + \frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2}\right) dt.$$

By a simple calculation we get that

$$\text{(21)} \quad \prod_{n=1}^{N} \exp\left\{\frac{\lambda^3 K^3 M_N a_n^2}{2}\right\} = \exp\left\{\frac{\lambda^3 K^3 M_N A_N^2}{2}\right\},$$

furthermore,

$$\int_0^1 \prod_{n=1}^{N} \left(1+\lambda a_n \varphi_n(t) + \frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2}\right) dt =$$

$$= 1 + \sum{}' \lambda^k a_{n_1} \cdots a_{n_k} \int_0^1 \varphi_{n_1}(t) \cdots \varphi_{n_k}(t)\, dt + \sum{}' \frac{\lambda^{2k}}{2^k} a_{n_1}^2 \cdots a_{n_k}^2 \int_0^1 \varphi_{n_1}^2(t) \cdots \varphi_{n_k}^2(t)\, dt +$$

$$+ \sum{}'' \frac{\lambda^{k+2l}}{2^l} a_{n_1} \cdots a_{n_k} a_{m_1}^2 \cdots a_{m_l}^2 \int_0^1 \varphi_{n_1}(t) \cdots \varphi_{n_k}(t) \varphi_{m_1}^2(t) \ldots \varphi_{m_l}^2(t)\, dt = 1 + I + J + K,$$

where the sum Σ' is extended for all systems of integer values $(1 \leqq)\, n_1 < \cdots < n_k (\leqq N)$ $(1 \leqq k \leqq N)$, the sum Σ'' is extended for all systems of integer values $(1 \leqq)\, n_1 < \cdots$ $\cdots < n_k\, (\leqq N)$ and $(1 \leqq)\, m_1 < \cdots < m_l\, (\leqq N)$ for which $n_i \neq m_j$ $(1 \leqq i \leqq k,\ 1 \leqq j \leqq l)$; $1 \leqq k$, $1 \leqq l$ and $k + l \leqq N$. It follows from (3) that $I = K = 0$ and

$$J = \sum{}' \frac{\lambda^2 a_{n_1}^2}{2} \cdots \frac{\lambda^2 a_{n_k}^2}{2}.$$

So we obtain that

$$\text{(22)} \quad \int_0^1 \prod_{n=1}^{N} \left(1+\lambda a_n \varphi_n(t) + \frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2}\right) dt = \prod_{n=1}^{N} \left(1 + \frac{\lambda^2 a_n^2}{2}\right).$$

Applying the well-known inequality

$$1 + u \leqq e^u \quad \text{if} \quad u \geqq 0,$$

from (20), (21) and (22) we get that

$$\int_0^1 \exp\{\lambda S_N(t)\}\,dt \leqq \exp\left\{\frac{\lambda^3 K^3 M_N A_N^2}{2}\right\} \prod_{n=1}^{N} \exp\left\{\frac{\lambda^2 a_n^2}{2}\right\} =$$

$$= \exp\left\{\frac{\lambda^3 K^3 M_N A_N^2}{2}\right\} \exp\left\{\frac{\lambda^2 A_N^2}{2}\right\} = \exp\left\{\frac{\lambda^2 A_N^2}{2}(1+\lambda M_N K^3)\right\}.$$

This shows that the right-hand inequality of (18) is true.

We get similarly to (20) that

$$\int_0^1 \exp\{\lambda S_N(t)\}\,dt \geqq \tag{23}$$

$$\geqq \prod_{n=1}^{N} \exp\left\{-\frac{\lambda^3 K^3 M_N a_n^2}{2}\right\} \int_0^1 \prod_{n=1}^{N} \left(1+\lambda a_n \varphi_n(t)+\frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2}\right) dt.$$

Applying the simple inequality

$$e^{u(1-u)} \leqq 1+u \quad \text{if} \quad u \geqq 0, \text{ [6]}$$

we get that

$$\prod_{n=1}^{N}\left(1+\frac{\lambda^2 a_n^2}{2}\right) \geqq \prod_{n=1}^{N} \exp\left\{\frac{\lambda^2 a_n^2}{2}\left(1-\frac{\lambda^2 M_N^2}{2}\right)\right\} = \exp\left\{\frac{\lambda^2 A_N^2}{2}\left(1-\frac{\lambda^2 M_N^2}{2}\right)\right\}.$$

This and (21), (22), (23) show that the left-hand inequality of (18) is true. This completes the proof of Lemma 1.

In the proof which follows we use some ideas from the classical paper of Kolmogoroff [2]. First we introduce the notation

$$W_N(x) = \text{mes}\,(\{S_N(t) \geqq x\}) \quad \text{for} \quad x > 0.$$

Proof of Theorem 1. Let λ be a positive real number determined later on. It is obvious that

$$W_N(x) e^{\lambda x} \leqq \int_0^1 \exp\{\lambda S_N(t)\}\,dt,$$

and it follows from (18) that

$$W_N(x) \leqq \exp\left\{-\lambda x + \frac{\lambda^2 A_N^2}{2}(1+\lambda M_N K^3)\right\}. \tag{24}$$

[6]) This sharper inequality $u-u^2/2 \leqq \log(1+u)$ $(u \geqq 0)$ is also true, as the function $\varkappa(u) = \log(1+u)-u+u^2/2$ is non-decreasing and $\varkappa(0)=0$.

Setting $\lambda = x/A_N^2$ we get

$$W_N(x) \leqq \exp\left\{-\frac{x^2}{A_N^2}+\frac{x^2}{2A_N^2}\left(1+\frac{xM_NK^3}{A_N^2}\right)\right\} = \exp\left\{-\frac{x^2}{2A_N^2}\left(1-\frac{xM_NK^3}{A_N^2}\right)\right\}.$$

This proves (4) and finishes the proof of Theorem 1.

We need the next two lemmas only for the proof of Theorem 2.

Lemma 2. *If*

$$\frac{xM_NK^3}{A_N^2} \leqq \frac{1}{2}, \tag{25}$$

then

$$W_N(x) \leqq \exp\left\{-\frac{x^2}{4A_N^2}\right\}. \tag{26}$$

Proof. As $\theta \leqq 1/2$ by (25), on the ground of Theorem 1, (26) holds obviously.

Lemma 3. *If*

$$\frac{xM_NK^3}{A_N^2} \geqq \frac{1}{2}, \tag{27}$$

then

$$W_N(x) \leqq \exp\left\{-\frac{x}{8M_NK^3}\right\}. \tag{28}$$

Proof. In the proof of Theorem 1 we obtained (24), where λ is an arbitrary positive real number. Now we set

$$\lambda = \frac{1}{2M_NK^3}.$$

From (24) and (27) we get that

$$W_N(x) \leqq \exp\left\{-\frac{x}{2M_NK^3}+\frac{A_N^2}{8M_N^2K^6}\left(1+\frac{1}{2}\right)\right\} \leqq$$

$$\leqq \exp\left\{-\frac{x}{2M_NK^3}+\frac{3x}{8M_NK^3}\right\} = \exp\left\{-\frac{x}{8M_NK^3}\right\}.$$

So the proof of Lemma 3 is ready.

The proof of the inequality (6) is much more involved. The following argument follows closely that of a similar theorem in the paper of KOLMOGOROFF [2].

Proof of Theorem 2. Let $\delta = \varepsilon/8$. Then

$$\delta^2 = \max(128\alpha, 16(\log\beta)/\beta). \tag{29}$$

Hence it follows that

$$\delta^2 \leqq 1/64, \quad \delta \leqq 1/8 \quad \text{and} \quad \delta > 2\delta^2. \tag{30}$$

We set now

$$\lambda = x/[A_N^2(1-\delta)]$$

so that, by (30),

$$x/A_N^2 < \lambda < 2x/A_N^2,$$

furthermore, in virtue of (5) we have

(31) $$\lambda M_N K^3 < 2\alpha \leqq 2^{-12}$$

and

(32) $$\lambda^2 A_N^2 > \beta \geqq 2^{14}.$$

On account of Lemma 1

$$\int_0^1 \exp\{\lambda S_N(t)\}\,dt \geqq \exp\{\tfrac{1}{2}\lambda^2 A_N^2(1-\tfrac{1}{2}\lambda^2 M_N^2 - \lambda M_N K^3)\}.$$

By (29) and (31), we get

$$\tfrac{1}{2}\lambda^2 M_N^2 + \lambda M_N K^3 < \tfrac{1}{2}(2\alpha)^2 + 2\alpha \leqq 4\alpha \leqq \delta^2/4.$$

Hence

(33) $$\int_0^1 \exp\{\lambda S_N(t)\}\,dt \geqq \exp\{\tfrac{1}{2}\lambda^2 A_N^2(1-\delta^2/4)\}.$$

On the other hand, integrating by parts, we obtain

$$\int_0^1 \exp\{\lambda S_N(t)\}\,dt = -\int_{-\infty}^{+\infty} e^{\lambda y}\,dW_N(y) = \lambda \int_{-\infty}^{+\infty} e^{\lambda y} W_N(y)\,dy.$$

We decompose the interval $(-\infty, +\infty)$ of integration into the five intervals $I_1 = (-\infty, 0]$, $I_2 = (0, \lambda A_N^2(1-\delta)]$, $I_3 = (\lambda A_N^2(1-\delta), \lambda A_N^2(1+\delta)]$, $I_4 = (\lambda A_N^2(1+\delta), 8\lambda A_N^2]$ and $I_5 = (8\lambda A_N^2, +\infty)$ and search for upper bounds of the integral over I_1 and I_5 and over I_2 and I_4.

We have

(34) $$J_1 = \lambda \int_{-\infty}^{0} e^{\lambda y} W_N(y)\,dy \leqq \lambda \int_{-\infty}^{0} e^{\lambda y}\,dy = 1$$

because $W_N(y) \leqq 1$ for all y. According to (31), Lemma 3, and Lemma 2, we have on I_5

$$W_N(y) \leqq \exp\left\{-\frac{y}{8M_N K^3}\right\} \leqq e^{-2\lambda y} \quad \text{for} \quad y \geqq \frac{A_N^2}{2M_N K^3},$$

and

$$W_N(y) \leqq \exp\left\{-\frac{y^2}{4A_N^2}\right\} \leqq e^{-2\lambda y} \quad \text{for} \quad 8\lambda A_N^2 \leqq y \leqq \frac{A_N^2}{2M_N K^3}.$$

Therefore

(35) $$J_5 = \lambda \int_{8\lambda A_N^2}^{+\infty} e^{\lambda y} W_N(y)\,dy \leqq \lambda \int_{8\lambda A_N^2}^{+\infty} e^{-\lambda y}\,dy < 1.$$

It follows, by (30), (32), and (33), that

$$\int_0^1 \exp\{\lambda S_N(t)\}\, dt > 8.$$

Hence, on account of (34) and (35), we can see that

$$J_1 + J_5 < \frac{1}{4}\int_0^1 \exp\{\lambda S_N(t)\}\, dt. \tag{36}$$

On the intervals I_2 and I_4, applying Theorem 1, we have

$$e^{\lambda y} W_N(y) \leqq \exp\left\{\lambda y - \frac{y^2}{2A_N^2}\left(1 - \frac{\delta^2}{8}\right)\right\} = e^{\mu(y)}$$

because, by (29) and (31), we obtain that

$$\theta = \frac{yM_N K^3}{A_N^2} \leqq 8\lambda M_N K^3 < 16\alpha \leqq \frac{\delta^2}{8}.$$

The quadratic expression $\mu(y)$ attains its maximum for $y = \lambda A_N^2(1-\delta^2/8)^{-1}$ which lies in I_3. Hence, in the intervals I_2 and I_4 $\mu(y)$ is majorized by its value at $y = \lambda A_N^2(1+\delta)$ (as $\lambda A_N^2(1+\delta)$ lies closer to the right endpoint of the interval I_3 than to the left one). This value does not exceed

$$\lambda^2 A_N^2(1+\delta) - \frac{\lambda^2 A_N^2}{2}(1+\delta)^2\left(1-\frac{\delta^2}{8}\right) =$$

$$= \frac{\lambda^2 A_N^2}{2}\left(1 - \delta^2 + \frac{\delta^2}{8}(1+\delta)^2\right) < \frac{\lambda^2 A_N^2}{2}\left(1 - \frac{\delta^2}{2}\right).$$

Therefore

$$J_2 + J_4 = \lambda\left\{\int_0^{\lambda A_N^2(1-\delta)} + \int_{\lambda A_N^2(1+\delta)}^{8\lambda A_N^2}\right\} e^{\lambda y} W_N(y)\, dy <$$

$$< \lambda \int_0^{8\lambda A_N^2} \exp\left\{\frac{\lambda^2 A_N^2}{2}\left(1-\frac{\delta^2}{2}\right)\right\} dy = 8\lambda^2 A_N^2 \exp\left\{\frac{\lambda^2 A_N^2}{2}\left(1-\frac{\delta^2}{2}\right)\right\}.$$

From (5), (29) and (32), we get the following estimates:

$$\log 2^7 \beta < 2\log\beta \leqq \frac{\beta\delta^2}{8},$$

(37)

$$\log 2^5 \lambda^2 A_N^2 < 2\log \lambda^2 A_N^2 \leqq \frac{\lambda^2 A_N^2}{8}\delta^2$$

because $\lambda^2 A_N^2 > \beta$ and $\log u/u$ is a decreasing function if $u \geqq e$. So we have from (33)

$$J_2 + J_4 < \frac{1}{4} \exp\left\{\frac{\lambda^2 A_N^2}{2}\left(1 - \frac{\delta^2}{4}\right)\right\} < \frac{1}{4}\int_0^1 \exp\{\lambda S_N(t)\}\, dt. \tag{38}$$

It follows, from (36) and (38)

$$J_3 = \int_{\lambda A_N^2(1-\delta)}^{\lambda A_N^2(1+\delta)} e^{\lambda y} w_N(y)\, dy > \frac{1}{2}\int_0^1 \exp\{\lambda S_N(t)\}\, dt > \frac{1}{2}\exp\left\{\frac{\lambda^2 A_N^2}{2}(1-\delta)\right\} \tag{39}$$

because $\delta > \delta^2/4$. Since $W_N(y)$ is a decreasing function, on account of the definition of λ, we have that

$$J_3 < 2\lambda^2 A_N^2 \delta \exp\{\lambda^2 A_N^2 (1+\delta)\} W_N(x). \tag{40}$$

From (39) and (40) we obtain that

$$W_N(x) > \frac{1}{4\lambda^2 A_N^2 \delta} \exp\left\{-\frac{\lambda^2 A_N^2}{2}(1+3\delta)\right\}.$$

Similarly to (37), we have

$$\log 4\lambda^2 A_N^2 \delta < \tfrac{1}{2}\lambda^2 A_N^2 \delta$$

as $4\lambda^2 A_N^2 \delta > 4\beta\delta \geqq 16\sqrt{\beta \log \beta} \geqq 2^{12}$, and $\log u/u \leqq 1/8$ if $u \geqq 2^{12}$. So we get that

$$W_N(x) \exp\left\{-\frac{\lambda^2 A_N^2}{2}(1+4\delta)\right\} = \exp\left\{-\frac{x^2}{2A_N^2(1-\delta)^2}(1+4\delta)\right\} >$$

$$> \exp\left\{-\frac{x^2}{2A_N^2}(1+8\delta)\right\} = \exp\left\{-\frac{x^2}{2A_N^2}(1+\varepsilon)\right\}$$

because $\delta = \varepsilon/8$ and, by (30), $\delta \leqq 1/8$. This yields (6) with a suitable ε, by (29). And this is what we wished to prove.

§ 2. The proof of Theorem 3 and Theorem 4

We need a result concerning series with RADEMACHER's functions defined as follows

$$r_n(x) = \operatorname{sign} \sin 2^{n+1}\pi x \qquad (0 \leqq x \leqq 1;\ n = 1, 2, \cdots).$$

The following assertion holds (see [4], v. 1, p. 213):

Lemma 4. *If p is an arbitrary positive real number then*

$$\left\{\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right\}^{\frac{1}{p}} \leqq 2p^{\frac{1}{2}} \left\{\sum_{n=1}^N a_n^2\right\}^{\frac{1}{2}}. \tag{41}$$

Proof of Theorem 3. This argument will follow closely that on page 215 of [4]. First we show (10), hence then the second inequality (8) immediately follows. The first inequality (8) follows from the second one by a simple argument.

Let K denote a common bound for the system $\{\varphi_n(t)\}$, i.e.

$$|\varphi_n(t)| \leqq K \qquad (0 \leqq t \leqq 1;\ n = 1, 2, \cdots).$$

Furthermore, let μ denote a sufficiently small positive real number. We set

$$S_N(t; x) = \sum_{n=1}^{N} a_n \varphi_n(t) r_n(x).$$

Applying (41), with a simple calculation we get

$$\int_0^1 \exp\{\mu S_N^2(t;x)\}\, dx = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \int_0^1 S_N^{2k}(t;x)\, dx \leqq \tag{42}$$

$$\leqq \sum_{k=0}^{\infty} \frac{k^k}{k!} \left\{4\mu \sum_{n=1}^{N} a_n^2 \varphi_n^2(t)\right\}^k \leqq \sum_{k=0}^{\infty} \left\{4\mu e \sum_{n=1}^{N} a_n^2 \varphi_n^2(t)\right\}^k$$

since $k^k/k! < \sum_{n=0}^{\infty} k^n/n! = e^k$. On the basis of (9i)

$$4e\mu \sum_{n=1}^{N} a_n^2 \varphi_n^2(t) \leqq 4e\mu K^2 A_N^2 \leqq \frac{1}{2}$$

if

$$\mu \leqq \frac{1}{8eK^2}. \tag{43}$$

Thus, the series on the right of (42) uniformly converges in t $(0 \leqq t \leqq 1)$, and its sum does not exceed 2.

Integrate (42) over $0 \leqq t \leqq 1$ and interchange the order of integration; then

$$\int_0^1 dx \int_0^1 \exp\{\mu S_N^2(t;x)\}\, dt \leqq 2.$$

It follows that there is a dyadic irrational [7]) number x_0 $(0 < x_0 < 1)$ for which

$$\int_0^1 \exp\{\mu S_N^2(t;x_0)\}\, dt \leqq 2. \tag{44}$$

Consider the following representation of $S_N(t)$

$$S_N(t) = K^2 \int_0^1 S_N(u; x_0) P_N(t, u; x_0)\, du, \tag{45}$$

[7]) x_0 is dyadic irrational number if $x_0 \neq p/2^q$ where p and q are positive natural numbers.

where

$$P_N(t,u;x_0) = \prod_{n=1}^{N}\left(1+\frac{\varphi_n(t)\varphi_n(u)r_n(x_0)}{K^2}\right).$$

First of all, $P_N(t, u; x_0)$ is non-negative. Furthermore, $P_N(t, u; x_0)$ is symmetric in t and u, and

$$\int_0^1 P_N(t,u;x_0)\,du = 1+\sum{}'\frac{1}{K^{2k}}\varphi_{n_1}(t)\cdots\varphi_{n_k}(t)r_{n_1}(x_0)\cdots r_{n_k}(x_0)\int_0^1 \varphi_{n_1}(u)\cdots\varphi_{n_k}(u)\,du,$$

where the sum Σ' is extended for all systems of integer values $(1\leqq)n_1<\cdots<n_k(\leqq N)$ $(1\leqq k\leqq N)$. It follows from (2) that

$$\int_0^1 P_N(t,u;x_0)\,du = 1. \tag{46}$$

As to the representation (45), after carrying out the multiplications and integrating term by term, the right-hand side can be written as follows:

$$K^2\sum_{n=1}^{N}a_n r_n(x_0)\int_0^1 \varphi_n(u)\,du+\sum_{n=1}^{N}\sum_{m=1}^{N}a_n r_n(x_0)r_m(x_0)\varphi_m(t)\int_0^1 \varphi_n(u)\varphi_m(u)\,du+$$

$$+\sum_{n=1}^{N}a_n r_n(x_0)\sum{}''\frac{1}{K^{2k-2}}\varphi_{n_1}(t)\cdots\varphi_{n_k}(t)r_{n_1}(x_0)\cdots r_{n_k}(x_0)\int_0^1 \varphi_n(u)\varphi_{n_1}(u)\cdots$$

$$\cdots\varphi_{n_k}(u)\,du = I+J+K,$$

where the sum Σ'' is extended for all systems of integer values $(1\leqq)n_1<\cdots<n_k(\leqq N)$ $(2\leqq k\leqq N)$. Taking into account that the functions $\varphi_n(t)$ are normed, it follows from (2) that $I=K=0$ and

$$J=\sum_{n=1}^{N}a_n r_n^2(x_0)\varphi_n(t) = S_N(t)$$

because $r_n^2(x_0)=1$ $(1\leqq n\leqq N)$. This proves (45).

The function $\chi(u)=\exp(\mu u^2)$ is increasing and convex for $u\geqq 0$. On account of (46), JENSEN's inequality (see [4], v. I, p. 24) gives

$$\chi\left(\frac{|S_N(t)|}{K^2}\right) = \chi\left(\int_0^1 |S_N(t;x_0)|\cdot P_N(t,u;x_0)\,du\right)\leqq$$

$$\leqq\int_0^1 \chi(|S_N(t;x_0)|)P_N(t,u;x_0)\,du.$$

Integrate this over $0 \leqq t \leqq 1$ and interchange the order of integration, from (44) and (46), we get that

$$\int_0^1 \chi\left(\frac{|S_N(t)|}{K^2}\right) dt \leqq \int_0^1 \chi(|S_N(t; x_0)|)\, du \int_0^1 P_N(t, u; x_0)\, dt =$$

$$= \int_0^1 \exp\{\mu S_N^2(u; x_0)\}\, du \leqq 2.$$

Now we set $\mu = K^4 \lambda$ then, it follows from (9ii), this μ satisfies (43). We finished the proof of (10).

As to the second inequality (8), we set

$$S_N^*(t) = S_N(t)/A_N.$$

The condition (9i) is satisfied by the coefficients of $S_N^*(t)$. Thus, if λ is sufficiently small then, on account of (10),

$$\int_0^1 \exp\{\lambda S_N^{*2}(t)\}\, dt = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^1 |S_N^*(t)|^{2k}\, dt \leqq 2.$$

Hence it follows for every k

$$\int_0^1 |S_N^*(t)|^{2k}\, dt \leqq \frac{2k!}{\lambda^k},$$

that is

$$\left\{\int_0^1 |S_N(t)|^{2k}\, dt\right\}^{\frac{1}{2k}} \leqq D_{2k} A_N,$$

where, choosing λ equal to $(8eK^6)^{-1}$,

$$D_{2k} = \{2 \cdot 8^k e^k K^{6k} k!\}^{\frac{1}{2k}} \leqq 8K^3 k^{\frac{1}{2}} \qquad (k = 1, 2, \cdots).$$

If now for the positive real number p we have $2k-2 \leqq p < 2k$ with a suitable natural number k then it is sufficient to remark that

$$\left\{\int_0^1 |S_N(t)|^p\, dt\right\}^{\frac{1}{p}} \leqq \left\{\int_0^1 |S_N(t)|^{2k} dt\right\}^{\frac{1}{2k}}$$

(see [4], v. I, p. 25).

It still remains to prove the first inequality (8). This is immediate for $p \geqq 2$, for then

$$\left\{\int_0^1 |S_N(t)|^p\, dt\right\}^{\frac{1}{p}} \geqq \left\{\int_0^1 S_N^2(t)\, dt\right\}^{\frac{1}{2}} = A_N.$$

If $0<p<2$, let α_1 and α_2 be positive and such that $\alpha_1+\alpha_2=1$, $2=p\alpha_1+4\alpha_2$. The function

$$\int_0^1 |S_N(t)|^\alpha \, dt$$

being logarithmically convex in α (see [4], v. I, p. 25),

$$A_N^2=\int_0^1 S_N^2(t)\,dt \leqq \left\{\int_0^1 |S_N(t)|^p\,dt\right\}^{\alpha_1}\left\{\int_0^1 S_N^4(t)\,dt\right\}^{\alpha_2} \leqq \left\{\int_0^1 |S_N(t)|^p\,dt\right\}^{\alpha_1}(D_4 A_N)^{\alpha_2},$$

which gives

$$\left\{\int_0^1 |S_N(t)|^p\,dt\right\}^{\frac{1}{p}} \geqq D_4^{-(4-2p)/p} A_N.$$

This completes the proof of Theorem 3.

The following lemma needs in the proof of Theorem 4.

Lemma 5. *There exist positive absolute constants η ($\geqq 1$) and ε such that*

$$\text{mes}\left(\{|S_N(t)|\geqq \eta A_N\}\right)\geqq \varepsilon.$$

The proof is based on a lemma which can find in [4], Chapter V, (8. 26), and it goes, applying (8), word by word as there.

As to the proof of Theorem 4, it can be proved in the same way as the analogous assertion for Rademacher functions, see [4], Chapter XV., (5. 14), applying (8) to prove the second inequality (11) and Lemma 5 the first one.

§ 3. The proof of Theorem 5

We are going to apply the following well-known assertion: if the sequence $\{E_k\}$ of measurable subsets of the interval [0, 1] is such that

$$\sum_{k=1}^{\infty} \text{mes}\,(E_k) < \infty,$$

then

$$\text{mes}\left(\limsup_{k\to\infty} E_k\right) = 0. \text{ [8]}$$

For the arbitrary fixed positive real number $\varepsilon(<1)$, we choose the real number $\eta(<1)$ such that

$$\eta(1+\varepsilon) > 1, \quad \text{e.g.} \quad \eta = 1-\frac{\varepsilon}{2}. \tag{47}$$

[8]) $\limsup\limits_{k\to\infty} E_k$ is the set of all those points which belong to infinitely many E_k.

Now we define the sequence of indices $n_1 \leqq n_2 \leqq \cdots$ in the following manner:

$$A^2_{n_k-1} \leqq e^{k^\eta} < A^2_{n_k} \qquad (k=1,2,\cdots). \tag{48}$$

This is possible in virtue of (14i).

We set

$$E_k = \left\{ \frac{S_{n_k}(t)}{\sqrt{2A^2_{n_k} \log\log A^2_{n_k}}} \geqq 1+\varepsilon \right\}.$$

On account of Theorem 1, we get that

$$\operatorname{mes}(E_k) = W_{n_k}\big((1+\varepsilon)\sqrt{2A^2_{n_k}\log\log A^2_{n_k}}\big) \leqq \exp\{-(1+\varepsilon)^2(1-\theta)\log\log A^2_{n_k}\}, \tag{49}$$

where

$$\theta = (1+\varepsilon)K^3 M_{n_k} \sqrt{\frac{2\log\log A^2_{n_k}}{A^2_{n_k}}}.$$

Here K denotes a common bound of the system $\{\varphi_n(t)\}$. Taking into account (14ii), this θ tends to 0 if k tends to ∞. Thus, θ is not greater than $\varepsilon/2$ if k is sufficiently large. Continuing the estimation (49) we obtain

$$\operatorname{mes}(E_k) \leqq \exp\left\{-(1+\varepsilon)^2\left(1-\frac{\varepsilon}{2}\right)\log\log A^2_{n_k}\right\} \leqq$$

$$\leqq \exp\{-(1+\varepsilon)\log\log A^2_{n_k}\} = (\log A^2_{n_k})^{-(1+\varepsilon)}.$$

By (48), hence we get

$$\sum_{k=1}^{\infty} \operatorname{mes}(E_k) \leqq \sum_{k=1}^{\infty} \frac{1}{k^{\eta(1+\varepsilon)}} < \infty$$

in virtue of (47). So, we have shown that in the case of sequence of indices defined by (48), we have that

$$\limsup_{k\to\infty} \frac{S_{n_k}(t)}{\sqrt{2A^2_{n_k}\log\log A^2_{n_k}}} \leqq 1+\varepsilon$$

holds almost everywhere.

Let $m_1 \leqq m_2 \leqq \cdots$ be an arbitrary sequence of indices for which

$$\begin{aligned} n_k \leqq m_k < n_{k+1} \quad &\text{if} \quad n_k \neq n_{k+1}, \quad \text{and} \\ n_k = m_k \quad &\text{if} \quad n_k = n_{k+1} \qquad (k=1,2,\cdots). \end{aligned} \tag{50}$$

It is sufficient to show that

$$\Delta_k(t) = \frac{S_{m_k}(t) - S_{n_k}(t)}{\sqrt{2A^2_{n_k}\log\log A^2_{n_k}}}$$

tends to 0 for almost every t $(0 \leqq t \leqq 1)$.

It is obvious that $\Delta_k(t)=0$ if $n_k = n_{k+1}$. Therefore, in the following we assume that $n_k < n_{k+1}$. Let p be a positive real number to be determined later on. Applying Theorem 3, we get

(51)
$$\int_0^1 \Delta_k^{2p}(t)\,dt \leqq D_{2p}^{2p}\left(\frac{A_{n_{k+1}-1}^2 - A_{n_k}^2}{2A_{n_k}^2 \log\log A_{n_k}^2}\right)^p \leqq$$
$$\leqq D_{2p}^{2p}\left(\frac{e^{(k+1)^\eta} - e^{k^\eta}}{2e^{k^\eta}\eta \log k}\right)^p \leqq D_{2p}^{2p}\left(\frac{e^{(k+1)^\eta - k^\eta} - 1}{2\eta}\right)^p.$$

We apply the following inequalities:

$$e^u \leqq 1 + 3u \quad \text{if} \quad 0 \leqq u \leqq 1,$$

and

$$(u+1)^\eta - u^\eta \leqq \frac{\eta}{u^{1-\eta}} \quad \text{if} \quad u \geqq 0 \qquad (0 < \eta < 1).$$ [9]

On account of these and (51), we obtain

$$\int_0^1 \Delta_k^{2p}(t)\,dt \leqq D_{2p}^{2p}\left(\frac{3((k+1)^\eta - k^\eta)}{2\eta}\right)^p \leqq \left(\frac{3}{2}\right)^p D_{2p}^{2p}\frac{1}{k^{p(1-\eta)}}.$$

If we fix the real number p so large that $p(1-\eta) > 1$ is satisfied then

$$\sum_{k=1}^{\infty}\int_0^1 \Delta_k^{2p}(t)\,dt \leqq \left(\frac{3}{2}\right)^p D_{2p}^{2p}\sum_{k=1}^{\infty}\frac{1}{k^{p(1-\eta)}} < \infty.$$

It follows from the theorem of Beppo Levi that

$$\Delta_k^{2p}(t) \to 0 \qquad (k \to \infty)$$

almost everywhere. As p is fixed, therefore we have proved the assertion (15).

Now, we set $N_1 = n_1$, and let N_l $(l \geqq 2)$ be equal to the first index n_k for which $n_k > N_{l-1}$. It is obvious that the sequence $N_1 < N_2 < \cdots$ has the property as asserted in Theorem 5.

§ 4. The proof of Theorem 6

Lemma 6. *Let $\{b_n\}$ be a sequence of non-negative real numbers. If*

(52)
$$\text{(i) } s_N = \sum_{n=1}^{N} b_n \to \infty \quad \textit{and} \quad \text{(ii) } b_N = o(s_N),$$

then for arbitrary positive real number $\alpha(>1)$, we have

(53)
$$\sum_{n=1}^{N} b_n^\alpha = o(s_N^\alpha).$$

[9]) This inequality follows from the fact that the function $u^\eta (0 < \eta < 1)$ is concave for $u \geqq 0$.

Proof. Let ε be an arbitrary positive real number. We choose the natural number n_0 in such a manner that

$$\frac{b_n}{s_n} \leqq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\alpha-1}} \quad \text{if} \quad n \geqq n_0.$$

This is possible in virtue of (52ii). Next we choose the natural number N_0 such that

$$\frac{1}{s_{N_0}} \sum_{n=1}^{n_0-1} b_n^\alpha \leqq \frac{\varepsilon}{2},$$

that is also possible in virtue of (52i). Then

$$\frac{1}{s_N^\alpha} \sum_{n=1}^{N} b_n^\alpha = \frac{1}{s_N^\alpha} \left\{\sum_{n=1}^{n_0-1} + \sum_{n=n_0}^{N}\right\} \leqq \frac{\varepsilon}{2} + \frac{1}{s_N^\alpha} \sum_{n=n_0}^{N} \frac{\varepsilon}{2} s_N^{\alpha-1} b_n \leqq \varepsilon$$

whenever $N \geqq N_0$, and assertion (53) is proved.

Proof of Theorem 6. [10]) In the proof we apply the following elementary inequality: for every real number u and every natural number n, we have (see [10], p. 365)

$$\left| e^{iu} - \sum_{k=0}^{n-1} \frac{(iu)^k}{k!} \right| \leqq \frac{|u|^n}{n!}. \tag{54}$$

We make use of the classical method of characteristic functions. Let us introduce the following notation:

$$\psi_N(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda y}\, dF_N(y),$$

where $F_N(y)$ is defined by (17). It is enough to prove that for any fixed λ the characteristic function $\psi_N(\lambda)$ tends to the characteristic function of the normal distribution, i.e.

$$\psi_N(\lambda) \to e^{-\frac{\lambda^2}{2}} \qquad (N \to \infty). \tag{55}$$

It is obvious that

$$\psi_N(\lambda) = \int_0^1 \exp\left\{\frac{i\lambda S_N(t)}{A_N}\right\} dt. \tag{56}$$

Applying (54) with $n=3$, we get that

$$\psi_N(\lambda) = \int_0^1 \prod_{n=1}^{N} \left\{\left(1 + \frac{i\lambda a_n \varphi_n(t)}{A_N} - \frac{\lambda^2 a_n^2 \varphi_n^2(t)}{2A_N^2}\right) + \theta_n \frac{\lambda^3 a_n^3 \varphi_n^3(t)}{6A_N^3}\right\} dt, \tag{57}$$

where θ_n also depends on N, and $|\theta_n| \leqq 1$ $(n=1, 2, \cdots, N)$.

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[10]) The proof follows that of LINDEBERG's theorem which is due to FELLER [9]. See also [10], pp. 365—368.

We show that the integral on the right-hand side of (57) can be replaced by the following simpler integral

$$\int_0^1 \prod_{n=1}^{N}\left(1+\frac{i\lambda a_n\varphi_n(t)}{A_N}-\frac{\lambda^2 a_n^2\varphi_n^2(t)}{2A_N^2}\right)dt, \tag{58}$$

in the sense that for every fixed λ the difference of (57) and (58) tends to 0 if N tends to ∞. For the sake of brevity we denote

$$P_n(t)=1+\frac{i\lambda a_n\varphi_n(t)}{A_N}-\frac{\lambda^2 a_n^2\varphi_n^2(t)}{2A_N^2} \quad \text{and} \quad R_n(t)=\theta_n\frac{\lambda^3 a_n^3\varphi_n^3(t)}{6A_N^3},$$

where we do not indicate the dependence on N. Applying the following identity (see [10], p. 367.)

$$\prod_{n=1}^{N}(p_n+r_n)-\prod_{n=1}^{N}p_n=\sum_{n=1}^{N}r_n\left(\prod_{k=1}^{n-1}p_k\right)\left(\prod_{k=n+1}^{N}(p_k+r_k)\right)$$

(the empty product equals 1), we obtain

$$\begin{aligned}\left|\exp\left\{\frac{i\lambda S_N(t)}{A_N}\right\}-\prod_{n=1}^{N}P_n(t)\right| &= \left|\prod_{n=1}^{N}(P_n(t)+R_n(t))-\prod_{n=1}^{N}P_n(t)\right| \leqq \\ &\leqq \sum_{n=1}^{N}|R_n(t)|\cdot\left(\prod_{k=1}^{n-1}|P_k(t)|\right)\cdot\left(\prod_{k=n+1}^{N}(|P_k(t)|+|R_k(t)|)\right).\end{aligned} \tag{59}$$

By a simple calculation, we get that

$$\begin{aligned}|P_n(t)| &= \left\{\left(1-\frac{\lambda^2 a_n^2\varphi_n^2(t)}{2A_N^2}\right)^2+\frac{\lambda^2 a_n^2\varphi_n^2(t)}{A_N^2}\right\}^{\frac12}= \\ &= \left\{1+\frac{\lambda^4 a_n^4\varphi_n^4(t)}{4A_N^4}\right\}^{\frac12}\leqq 1+\frac{\lambda^2 a_n^2 K^2}{2A_N^2},\end{aligned} \tag{60}$$

furthermore,

$$|R_n(t)|\leqq\frac{|\lambda|^3|a_n|^3K^3}{6A_N^3} \qquad (n=1,2,\dots,N), \tag{61}$$

where K denotes a common bound of the system $\{\varphi_n(t)\}$.

From (60) and (61) we obtain that the right-hand side of (59) does not exceed

$$\sum_{n=1}^{N}\frac{|\lambda|^3K^3|a_n|^3}{6A_N^3}\left\{\prod_{k=1}^{n-1}\left(1+\frac{\lambda^2a_k^2K^2}{2A_N^2}\right)\prod_{k=n+1}^{N}\left(1+\frac{\lambda^2a_k^2K^2}{2A_N^2}+\frac{|\lambda|^3K^3|a_k|^3}{6A_N^3}\right)\right\}.$$

Applying the inequality $1+u\leqq e^u$ $(u\geqq 0)$, the last sum is not greater than

$$\sum_{n=1}^{N}\frac{|\lambda|^3K^3|a_n|^3}{6A_N^3}\exp\left\{\sum_{k=1}^{N}\frac{\lambda^2K^2a_k^2}{2A_N^2}+\sum_{k=n+1}^{N}\frac{|\lambda|^3K^3|a_k|^3}{6A_N^3}\right\}\leqq$$

$$\leqq\frac{|\lambda|^3K^3}{6}\cdot\exp\left\{\frac{\lambda^2K^2}{2}+\frac{|\lambda|^3K^3}{6}\right\}\cdot\frac{1}{A_N^3}\sum_{n=1}^{N}|a_n|^3,$$

as it is clear that $A_N^{-3}\sum_{n=1}^{\infty}|a_n|^3\leqq 1$. It follows from (16) that the conditions (52) of Lemma 6 are satisfied by the sequence $\{a_n^2\}$. Therefore, applying Lemma 6 with $\alpha=3/2$, on the basis of (53), we get that the difference of the integrand of (57) and (58) tends to 0 $(N\to\infty)$ uniformly in t $(0\leqq t\leqq 1)$ if λ is fixed.

To prove (55) for any fixed λ, we need the following inequalities:

(62)
$$1-u\leqq e^{-u}\quad\text{if}\quad u\geqq 0,$$
$$e^{-u(1+u)}\leqq 1-u\quad\text{if}\quad 0\leqq u\leqq\frac{\sqrt{2}-1}{2}.\ ^{11)}$$

Now carry out the multiplication in the integrand of (58) and integrate term by term

$$\int_0^1\prod_{n=1}^{N}\left(1+\frac{i\lambda a_n\varphi_n(t)}{A_N}-\frac{\lambda^2a_n^2\varphi_n^2(t)}{2A_N^2}\right)dt=$$

$$=1+\sum{}'\frac{i^k\lambda^k}{A_N^k}a_{n_1}\cdots a_{n_k}\int_0^1\varphi_{n_1}(t)\cdots\varphi_{n_k}(t)\,dt+$$

$$+\sum{}'\frac{(-1)^k\lambda^{2k}}{2^kA_N^{2k}}a_{n_1}^2\cdots a_{n_k}^2\int_0^1\varphi_{n_1}^2(t)\cdots\varphi_{n_k}^2(t)\,dt+$$

$$+\sum{}''\frac{i^k(-1)^l\lambda^{k+2l}}{2^kA_N^{k+2l}}a_{n_1}\cdots a_{n_k}a_{m_1}^2\cdots a_{m_l}^2\int_0^1\varphi_{n_1}(t)\cdots\varphi_{n_k}(t)\varphi_{m_1}^2(t)\cdots\varphi_{m_l}^2(t)\,dt,$$

where the sum Σ' is extended for all systems of integer values $(1\leqq)\ n_1<\cdots<n_k(\leqq N)$ $(1\leqq k\leqq N)$, the sum Σ'' is extended for all systems of integer values $(1\leqq)\ n_1<\cdots<n_k\,(\leqq N)$ and $(1\leqq)\ m_1<\cdots<m_l(\leqq N)$ for which $n_i\neq m_j$ $(1\leqq i\leqq k,\ 1\leqq j\leqq l)$; $1\leqq k$, $1\leqq l$ and $k+l\leqq N$. It follows from (3) that the integral (58) equals

$$1+\sum{}'\frac{(-1)^k\lambda^{2k}}{2^kA_N^{2k}}a_{n_1}^2\cdots a_{n_k}^2=\prod_{n=1}^{N}\left(1-\frac{\lambda^2a_n^2}{2A_N^2}\right).$$

[11]) This inequality follows from the fact that the curve $v=e^{-u(1+u)}$ $\left((-1-\sqrt{2})/2\leqq u\leqq (-1+\sqrt{2})/2\right)$, which is concave, lies below its tangent at the point $u=0$, $v=1$.

Taking into account (62), on the one hand

$$\prod_{n=1}^{N}\left(1-\frac{\lambda^2 a_n^2}{2A_N^2}\right) \leqq \exp\left\{-\sum_{n=1}^{N}\frac{\lambda^2 a_n^2}{2A_N^2}\right\} = e^{-\frac{\lambda^2}{2}} \tag{63}$$

holds for every N, on the other hand

$$\prod_{n=1}^{N}\left(1-\frac{\lambda^2 a_n^2}{2A_N^2}\right) \geqq \exp\left\{-\sum_{n=1}^{N}\frac{\lambda^2 a_n^2}{2A_N^2}\left(1+\frac{\lambda^2 a_n^2}{2A_N^2}\right)\right\} = \exp\left\{-\frac{\lambda^2}{2}-\sum_{n=1}^{N}\frac{\lambda^4 a_n^4}{4A_N^4}\right\} \tag{64}$$

holds if

$$\frac{\lambda^2 a_n^2}{2A_N^2} \leqq \frac{\sqrt{2}-1}{2} \qquad (1 \leqq n \leqq N).$$

But, in virtue of (16ii), this is satisfied for every sufficiently large N. Applying again Lemma 6 with $\alpha=2$, we get

$$\sum_{n=1}^{N}\frac{\lambda^4 a_n^4}{4A_N^4} \to 0 \qquad (N\to\infty).$$

According to (63) and (64)

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left(1-\frac{\lambda^2 a_n^2}{2A_N^2}\right) = e^{-\frac{\lambda^2}{2}}$$

holds for every fixed λ. This completes the proof of Theorem 6.

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