Inequalities for polynomials and their derivatives

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Introduction

1. Recently J. BALÁZS and P. TURÁN [2] have obtained certain interesting inequalities which arise from their consideration of (0, 2) interpolation on π-abscissas ($\pi_n(x)=(1-x^2)P'_{n-1}(x)$, $P_n(x)$ being the Legendre polynomial of degree $\leqq n$). By (0, 2) interpolation they mean the problem of finding interpolatory polynomials $R_n(x)$ of degree $\leqq 2n-1$ for which

(1. 1. 1) $$R_n(x_k)=\alpha_k, \quad R_n''(x_k)=\beta_k \qquad (k=1,2,\cdots,n)$$

are prescribed. From this consideration they proved the following

Theorem 1. 1. 1. *Let n be even and further if we are given for a polynomial $Q_{2n-1}(x)$ of degree $\leqq 2n-1$*

(1. 1. 2) $$|Q_{2n-1}(x_k)|\leqq A, \quad |Q''_{2n-1}(x_k)|\leqq B \qquad (k=1,2,\cdots,n)$$

then for $-1\leqq x\leqq +1$ *we have*

(1. 1. 3) $$|Q_{2n-1}(x)|\leqq \pi^6 nA+\frac{\pi^5 B}{n}$$

and

(1. 1. 4) $$|Q'_{2n-1}(x)|\leqq \pi^8 n^{5/2}A+\pi^5 Bn^{1/2}.$$

2. The appearence of the exponent 5/2 in (1. 1. 4) is unusual. They proved that the results (1. 1. 3) and (1. 1. 4) are also best possible in a certain sense. The object of this note is to obtain analogous results when the x_k's are taken to be the zeros of $(1-x^2)T_n(x)$, $T_n(x)$ being the Tchebycheff polynomials of the first kind.

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In an earlier work [3] we proved that (for n even) there exists a unique polynomial $R_n(x)$ of degree $\leqq 2n+1$ for which

$$R_n(x_k) = a_k \qquad (k = 1, 2, \cdots, n+2), \tag{1.2.1}$$

$$R_n''(x_k) = b_k \qquad (k = 2, 3, \cdots, n+1) \tag{1.2.2}$$

are prescribed in advance. Let

$$1 = x_1 > x_2 > \cdots > x_{n+2} = -1 \tag{1.2.3}$$

be the zeros of the polynomial $(1-x^2)T_n(x)$, $T_n(x) = \cos(n \arccos x)$. From our earlier work [3] we have

$$R_n(x) = \sum_{k=1}^{n+2} a_k r_k(x) + \sum_{k=2}^{n+1} b_k \varrho_k(x) \quad (n \text{ even}), \tag{1.2.4}$$

where fundamental polynomials $r_k(x)$ and $\varrho_k(x)$ are mentioned in the next section. From the uniqueness theorem [3] it follows that if $Q_{2n+1}(x)$ is an arbitrary polynomial of degree $\leqq 2n+1$, then

$$Q_{2n+1}(x) = \sum_{k=1}^{n+2} Q_{2n+1}(x_k) r_k(x) + \sum_{k=2}^{n+1} Q_{2n+1}''(x_k) \varrho_k(x). \tag{1.2.5}$$

Based on this we shall prove the following main theorem:

Theorem 1.2.1. *Suppose the polynomial $Q_{2n+1}(x)$ of degree $\leqq 2n+1$ (n even) satisfies:*

$$|Q_{2n+1}(x_k)| \leqq A \qquad (k = 1, 2, \cdots, n+2), \tag{1.2.6}$$

$$|Q_{2n+1}''(x_k)| \leqq \frac{B}{1-x_k^2} \qquad (k = 2, 3, \cdots, n+1). \tag{1.2.7}$$

Then for $-1 \leqq x \leqq +1$ we have

$$|Q_{2n+1}(x)| \leqq c_1(n^{3/2}A + Bn^{-1/2}) \quad \textit{with} \quad (c_1 = 54) \tag{1.2.8}$$

and

$$|Q_{2n+1}'(x)| \leqq c_2(n^{5/2}A + Bn^{1/2}) \quad \textit{with} \quad c_2 = 251. \tag{1.2.9}$$

First we remark that the result (1.2.8) is essentially best possible, i.e. we can find a suitable polynomial $f_0(x)$ of degree $\leqq 2n+1$ which satisfies (1.2.6) and (1.2.7) and for a numerical positive c_3

$$|f_0(d_n)| > c_3(An^{3/2} + Bn^{-1/2}) \tag{1.2.10}$$

where $d_n = \cos\chi_n$, $\chi_n = \dfrac{\pi}{2} - \dfrac{\pi}{4n}$. Thus comparing the results on these two abscissas we find that (1.2.8) is not so good as (1.1.3) although (1.2.8) is best possible as explained above. Nevertheless, the estimation of the derivative in both cases are equally good. If we apply MARKOV's inequality on (1. 2. 8) in the closed interval $-1 \leqq x \leqq +1$ we get

(1. 2. 11) $$|Q'_{2n+1}(x)| \leqq c_1(An^{7/2}+Bn^{3/2}).$$

The result stated in (1. 2. 9) is much better than (1. 2. 11). If, however, we consider only closed subintervals of $(-1, 1)$, S. BERNSTEIN's inequality gives from (1. 2. 8) that

(1. 2. 12) $$|Q'_{2n+1}(x)| \leqq \frac{2c_1}{\sqrt{\varepsilon}}[An^{5/2}+Bn^{1/2}] \quad \text{for} \quad -1+\varepsilon \leqq x \leqq 1-\varepsilon.$$

Comparing (1. 2. 9) with (1. 2. 12) we observe that both inequalities assert in $-1+\varepsilon \leqq x \leqq 1-\varepsilon$ essentially the same thing.

2. Preliminaries

1. The explicit forms of the fundamental functions $r_k(x)$ and $\varrho_k(x)$ $(k=2, 3, \cdots, n+1)$ that we have obtained in [3] are the following:

(2. 1. 1) $$\varrho_k(x) = \frac{(1-x^2)^{1/4}T_n(x)}{2T'_n(x_k)}\left[A_k\int_{-1}^{x}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{x}\frac{l_k(t)}{(1-t^2)^{1/4}}\,dt\right]$$

where

(2. 1. 2) $$A_k\int_{-1}^{+1}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt = -\int_{-1}^{+1}\frac{l_k(t)}{(1-t^2)^{1/4}}\,dt$$

and $l_k(t)$ is the fundamental polynomial of Lagrange interpolation

(2. 1. 3) $$l_k(t) = \frac{T_n(t)}{(t-x_k)T'_n(x_k)} \qquad (k = 2, 3, \cdots, n+1),$$

(2. 1. 4) $$r_k(x) = \frac{(1-x^2)}{2(1-x_k^2)}\,l_k^2(x) + \frac{(1-x^2)l_k(x)T'_n(x)}{2(1-x_k^2)T'_n(x_k)} + b_k\varrho_k(x) + \\ + \frac{(1-x^2)^{1/4}T_n(x)}{4(1-x_k^2)T'_n(x_k)}\left[A'_k\int_{-1}^{x}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{x}\frac{tl'_k(t)}{(1-t^2)^{1/4}}\,dt\right],$$

where

(2. 1. 5) $$b_k = \frac{n^2}{1-x_k^2} + \frac{1}{(1-x_k^2)^2},$$

(2. 1. 6) $$A'_k\int_{-1}^{+1}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt = -\int_{-1}^{+1}\frac{tl'_k(t)}{(1-t^2)^{1/4}}\,dt.$$

For $k=1$ and $k=n+2$ we have

(2.1.7) $$r_1(x) = \frac{1+x}{2} T_n^2(x) + (1-x^2)T_n(x)T_n'(x) - \frac{(1-x^2)^{1/4}T_n(x)}{2}\int_{-1}^{x} \frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt$$

and

(2.1.8) $$r_{n+2}(x) = \frac{1-x}{2} T_n^2(x) - (1-x^2)T_n(x)T_n'(x) - \frac{(1-x^2)^{1/4}T_n(x)}{2}\int_{-1}^{x} \frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt.$$

2. We wish to express these fundamental polynomials in another form, suitable to our purpose. For this we denote

(2. 2. 1) $$P_{2r}(x) = \frac{-\Gamma\left(r-\frac{1}{4}\right)}{\Gamma\left(r+\frac{5}{4}\right)} \sum_{i=0}^{r-1} \frac{\Gamma\left(i+\frac{5}{4}\right)}{\Gamma\left(i+\frac{3}{4}\right)} T_{2i+1}(x) = \text{Polynomial part of } (1-x^2)^{-3/4}\int_{-1}^{x} \frac{T_{2r}(t)}{(1-t^2)^{1/4}}$$

and

(2. 2. 2) $$V_{2r-1}(x) = \frac{-\Gamma\left(r-\frac{3}{4}\right)}{\Gamma\left(r+\frac{3}{4}\right)} \left[\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \sum_{i=1}^{r-1} \frac{\Gamma\left(i+\frac{3}{4}\right)}{\Gamma\left(i+\frac{1}{4}\right)} T_{2i}(x)\right] = (1-x^2)^{-3/4}\int_{-1}^{x} \frac{T_{2r}(t)\,dt}{(1-t^2)^{1/4}}.$$

Thus $\varrho_k(x)$ and $r_k(x)$ can be written, for $k=2, 3, \cdots, n+1$, in the following forms

(2. 2. 3) $$\varrho_k(x) = \frac{(1-x^2)T_n(x)}{2T_n'(x_k)} q_{n-1}(x)$$

where

(2. 2. 4) $$q_{n-1}(x) = A_k P_n(x) + \frac{2}{n}\sum_{r=1}^{\frac{n}{2}} [T_{2r}(x_k)P_{2r}(x) + T_{2r-1}(x_k)V_{2r-1}(x)]$$

and A_k is defined by (2. 1. 2), further

(2. 2. 5) $$r_k(x) = \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + b_k\varrho_k(x) + \frac{(1-x^2)T_n(x)s_{n-1}(x)}{4T_n'(x_k)(1-x_k^2)}$$

where $s_{n-1}(x)$ is given by

(2.2.6) $$s_{n-1}(x) = A'_k P_n(x) - \frac{4}{n} \sum_{r=1}^{\frac{n}{2}} \left(4r^2 T_{2r}(x_k) P_{2r}(x) + (2r-1)^2 T_{2r-1}(x_k) V_{2r-1}(x)\right).$$

Here b_k and A'_k are stated in (2. 1. 5) and (2. 1. 6).

3. We will prove some results which we require in the estimation of the fundamental polynomials.

Lemma 2. 3. 1. *For $k=2, 3, \cdots, n+1$ we have*

(2. 3. 1) $$\int_{-1}^{+1} \frac{l_k(t)}{(1-t^2)^{1/4}}\, dt \geqq 0,$$

(2. 3. 2) $$\sum_{k=2}^{n+1} \frac{1}{\sin \theta_k} \int_{-1}^{+1} \frac{l_k(t)}{(1-t^2)^{1/4}}\, dt \leqq 12.$$

This lemma is established in our earlier work [see formula (5. 8) and (4. 7) in [4]].

Lemma 2. 3. 2. *For $k=2, 3, \cdots, n+1$ we have*

(2. 3. 3) $$|d_k| \equiv \left| \int_{-1}^{+1} \frac{t\, l'_k(t)}{(1-t^2)^{1/4}}\, dt \right| \leqq \frac{4}{\sqrt{n}} \frac{1}{\sqrt{1-x_k^2}}.$$

Proof. From a result of L. Fejér we have

$$l'_k(t) = \frac{2}{n} \sum_{r=1}^{n-1} T_r(x_k) T'_r(t).$$

Integration by parts and using the differential equation for $T_r(t)$ yields

$$\int_{-1}^{+1} \frac{t T'_r(t)}{(1-t^2)^{1/4}}\, dt = \frac{\sqrt{\pi}}{2} r^2 \frac{\Gamma\left(\frac{r}{2} - \frac{1}{4}\right)}{\Gamma\left(\frac{r}{2} + \frac{5}{4}\right)} \text{ for } r \text{ even}, = 0 \text{ for } r \text{ odd}.$$

From this the result follows by using Abel's inequality.

Lemma 2. 3. 3. *For $-1 \leqq x \leqq +1$ we have*

(2. 3. 4) $$|P_{2r}(x)| \leqq 1, \quad |V_{2r-1}(x)| \leqq 1,$$

(2. 3. 5) $$|(1-x^2)^{1/2} P_{2r}(x)| \leqq \frac{1}{r}, \quad |(1-x^2)^{1/2} V_{2r-1}(x)| \leqq \frac{1}{r} \qquad (r \geqq 1),$$

(2. 3. 6) $$|(1-x^2) P'_{2r}(x)| \leqq 2, \quad |(1-x^2) V'_{2r-1}(x)| \leqq 2,$$

where $P_{2r}(x)$ and $V_{2r-1}(x)$ are defined in (2. 2. 1) *and* (2. 2. 2), *respectively.*

Proof. We will prove results for $P_{2r}(x)$, the corresponding results for $V_{2r-1}(x)$ are similar. From (2. 2. 1) we have

$$|P_{2r}(x)| \leqq r \frac{\Gamma\left(r-\frac{1}{4}\right)}{\Gamma\left(r+\frac{5}{4}\right)} \frac{\Gamma\left(r+\frac{1}{4}\right)}{\Gamma\left(r-\frac{1}{4}\right)} \leqq 1.$$

In order to prove (2. 3. 5) we observe that $\dfrac{\Gamma\left(i+\frac{5}{4}\right)}{\Gamma\left(i+\frac{3}{4}\right)}$ is a monotonically increasing function of i. Using ABEL's inequality we get

$$|(1-x^2)^{1/2} P_{2r}(x)| \leqq \frac{\Gamma\left(r-\frac{1}{4}\right)}{\Gamma\left(r+\frac{5}{4}\right)} \frac{\Gamma\left(r+\frac{1}{4}\right)}{\Gamma\left(r-\frac{1}{4}\right)} \max_{1\leqq p\leqq r-1} \left|\sum_{i=1}^{p} \cos(2i+1)\theta \sin\theta\right| \leqq \frac{1}{r}.$$

Again, using ABEL's inequality, we have

$$|(1-x^2)P_{2r}'(x)| \leqq \left|\frac{(2r-1)}{r+\frac{1}{4}} \max_{1\leqq p\leqq r-1} \sum_{i=1}^{p} \sin(2i+1)\theta \sin\theta\right| \leqq 2.$$

This completes the proof of the above Lemma by using the representation of $q_{n-1}(x)$ as given in (2. 2. 4). From the above lemma we get

Lemma 2. 3. 4. *For* $-1\leqq x\leqq +1$ *we have*

(2. 3. 7) $$|q_{n-1}(x)| \leqq 4+2C_k n^{3/2};$$

(2. 3. 8) $$|(1-x^2)^{1/2} q_{n-1}(x)| \leqq \frac{4}{n}\log n + 2n^{1/2} C_k,$$

(2. 3. 9) $$|(1-x^2) q_{n-1}'(x)| \leqq 8+4n^{3/2} C_k.$$

Here C_k *is given by*

(2. 3. 10) $$C_k = \int_{-1}^{+1} \frac{l_k(t)}{(1-t^2)^{1/4}}\, dt.$$

Let us denote

$$t_1(x) = (1-x^2) T_n(x) q_{n-1}(x).$$

Then from Lemma 2. 3. 4 we have at once

(2. 3. 11) $$|t_1(x)| \leqq \frac{4}{n}\log n + 2n^{1/2}C_k \qquad (-1 \leqq x \leqq +1),$$

(2. 3. 12) $$|t_1'(x)| \leqq 20\log n + 10n^{3/2}C_k \qquad (-1 \leqq x \leqq +1).$$

3. Estimates of the fundamental polynomials

1. The above Lemmas lead us to formulate:

Lemma 3. 1. 1. *For* $-1 \leqq x \leqq +1$ *we have*

(3. 1. 1) $$\sum_{k=2}^{n+1} |\varrho_k(x)| \leqq \sum_{k=2}^{n+1} \frac{|\varrho_k(x)|}{(1-x_k^2)} \leqq \frac{14}{n^{1/2}},$$

(3. 1. 2) $$\sum_{k=2}^{n+1} |\varrho_k'(x)| \leqq \sum_{k=2}^{n+1} \frac{|\varrho_k'(x)|}{1-x_k^2} \leqq 80n^{1/2}.$$

From (2. 2. 3), (2. 3. 10), (2. 3. 1), (2. 3. 2) and (2. 3. 11) we have

$$\sum_{k=2}^{n+1} \frac{|\varrho_k(x)|}{1-x_k^2} \leqq \frac{1}{2n}\left[\frac{4}{n}\log n \sum_{k=2}^{n+1} \frac{1}{\sqrt{1-x_k^2}} + 2 \cdot n^{1/2} \cdot 12\right],$$

Using again the above relations and (2. 3. 12) we have

$$\sum_{k=2}^{n+1} \frac{|\varrho_k'(x)|}{1-x_k^2} \leqq \frac{1}{2n}\left[20\log n \sum_{k=2}^{n+1} \frac{1}{\sqrt{1-x_k^2}} + 10n^{3/2} \cdot 12\right] \leqq 80n^{1/2}.$$

2. In order to determine the estimate of the fundamental polynomials of the first kind we need the following Lemmas:

Lemma 3. 2. 1. *For* $-1 \leqq x \leqq +1$ *we have*

(3. 2. 1) $$\sum_{k=2}^{n+1} \frac{(1-x^2)l_k^2(x)}{1-x_k^2} \leqq 8,$$

(3. 2. 2) $$\sum_{k=2}^{n+1} \left[\frac{(1-x^2)l_k^2(x)}{1-x_k^2}\right]' \leqq 36n^2,$$

where dash denotes differentiation with respect to x.

A proof of (3. 2. 1) is given in our earlier work [4], and (3. 2. 2) follows very easily by using the inequalities:

(3. 2. 3) $$|l_k(x)| \leqq 2 \qquad (-1 \leqq x \leqq +1),$$

(3. 2. 4) $$|(1-x^2)^{1/2} l_k'(x)| \leqq 2n \qquad (-1 \leqq x \leqq +1).$$

Lemma 3. 2. 2. *For* $-1\leqq x+1$ *we have*

(3. 2. 5) $$|s_{n-1}(x)|\leqq 3n^2+2n^{3/2}|d_k|,$$

(3. 2. 6) $$|(1-x^2)^{1/2}s_{n-1}(x)|\leqq 16n+2n^{1/2}|d_k|,$$

(3. 2. 7) $$|(1-x^2)s'_{n-1}(x)|\leqq 1n^2+4n^{3/2}|d_k|,$$

where d_k *is defined by* (2. 3. 3), *and* $s_{n-1}(x)$ *is a polynomial in* x *of degree* $\leqq n-1$ *given by* (2. 2. 6).

The proof of this lemma is clear from Lemma 2. 3. 3 and (2. 1. 6), and so we omit the details. Let us denote:

(3. 2. 8) $$t_2(x)=(1-x^2)T_n(x)s_{n-1}(x);$$

then by the above Lemma 3. 2. 2 it follows that

(3. 2. 9) $$|t_2(x)|\leqq 16n+2n^{1/2}|d_k| \qquad (-1\leqq x\leqq +1),$$

(3. 2. 10) $$|t'_2(x)|\leqq 33n^2+10n^{3/2}|d_k| \qquad (-1\leqq x\leqq +1).$$

3. Next we state:

Lemma 3. 3. 1. *For* $-1\leqq x\leqq +1$ *we have*

(3. 3. 1) $$|r_1(x)|\leqq 3n, \quad |r_{n+2}(x)|\leqq 3n,$$

(3. 3. 2) $$|r'_1(x)|\leqq 13n^2, \quad |r'_{n+2}(x)|\leqq 13n^2.$$

A proof of (3. 3. 1) is given in our earlier work [formula 6. 10, [4]] and (3. 3. 2) can be obtained easily by a simple computation using similar ideas as in Lemma 2. 3. 3.

Lemma 3. 3. 2. *For* $-1\leqq x\leqq +1$, *we have*

(3. 3. 3) $$\sum_{k=1}^{n+2}|r_k(x)|\leqq C_5 n^{3/2} \quad \text{with} \quad C_5=54$$

and

(3. 3. 4) $$\sum_{k=1}^{n+2}|r'_k(x)|\leqq C_6 n^{5/2} \quad \text{with} \quad C_6=251.$$

Proof. Using the representation of $r_k(x)$ as given in (2. 2. 5) we have

$$\sum_{k=2}^{n+1}|r_k(x)|\leqq 8+2n^2\sum_{k=2}^{n+1}\frac{|\varrho_k(x)|}{(1-x_k^2)}+\sum_{k=2}^{n+1}\frac{2n^{1/2}|d_k|+16n}{4n\sqrt{1-x_k^2}}\leqq$$

$$\leqq 8+2n^2\frac{14}{n^{1/2}}+4n\log n+4n\sum_{k=2}^{n+1}\frac{1}{k^2}\leqq 48n^{3/2}.$$

Here we have used (2. 3. 3), (3. 2. 1), (3. 1. 1). Combining with (3. 3. 1) we obtain (3. 3. 3) with $C_5=48$. Again,

$$\sum_{k=2}^{n+1}{}' |r_k'(x)| \leqq 36n^2+2n^2 \sum_{k=2}^{n+1}{}' \frac{|\varrho_k'(x)|}{(1-x_k^2)} + \sum_{k=2}^{n+1}{}' \frac{33n^2+10n^{3/2}|d_k|}{4n\sqrt{1-x_k^2}} \leqq$$

$$\leqq 36n^2+2n^2\cdot 80n^{1/2}+9n^2\log n+20n^2 \leqq 225n^{5/2}.$$

Here we have used (2. 3. 3), (3. 2. 2) and (3. 1. 2). Combining with (3. 3. 2) we obtain 3. 3. 4.

Proof of Theorem 1. 2. 1. From the representation of $Q_{2n+1}(x)$ as given in (1. 2. 5) we have on using (1. 2. 6), (1. 2. 7):

$$|Q_{2n+1}(x)| \leqq A\sum_{k=1}^{n+2} |r_k(x)| + B\sum_{k=2}^{n+1}{}' \frac{|\varrho_k(x)|}{1-x_k^2} \leqq C_1(An^{3/2}+Bn^{1/2}).$$

Here we have used only Lemma 3. 1. 1 and Lemma 3. 3. 2. Similarly using the same Lemmas

$$|Q_{2n+1}'(x)| \leqq A\sum_{k=1}^{n+2} |r_k'(x)| + B\sum_{k=2}^{n+1}{}' \frac{|\varrho_k'(x)|}{1-x_k^2} \leqq C_2(An^{5/2}+Bn^{1/2}).$$

Now it remains to prove (1. 2. 10). From our earlier work [5. 3, 6. 9 [4]] we know

$$\sum_{k=2}^{n+1}{}' \frac{|\varrho_k(d_n)|}{1-x_k^2} \geqq n^{-1/2}, \tag{3. 3. 5}$$

$$\sum_{k=1}^{n+2} |r_k(d_n)| \geqq 2^{-10}n^{3/2}, \tag{3. 3. 6}$$

where $d_n=\cos\chi_n$, $\chi_n=\frac{\pi}{2}-\frac{\pi}{4n}$. The polynomial $f_0(x)$ stated in (1. 2. 10) has the following representation:

$$f_0(x) = \sum_{k=1}^{n+2} A\operatorname{sign} r_k(d_n) r_k(x) + \sum_{k=2}^{n+1} B\varrho_k(x)\cdot(1-x_k^2)^{-1}\operatorname{sign}\varrho_k(d_n).$$

Obviously,

$$f_0(x_k) = A\operatorname{sign} r_k(d_n), \quad f_0''(x_k) = B(1-x_k^2)^{-1}\cdot\operatorname{sign}\varrho_k(d_n).$$

Therefore

$$f_0(d_n) = A\sum_{k=1}^{n+2} |r_k(d_n)| + B\sum_{k=2}^{n+1}{}' \frac{|\varrho_k(d_n)|}{1-x_k^2} \geqq C_3(An^{3/2}+Bn^{-1/2})$$

from (3. 3. 5) and (3. 3. 6). This completes the proof of the theorem.

Note. It is rather easy to prove that

$$\sum_{k=0}^{n+2} |r_k'(0)| \geqq C_5 n^{5/2} \quad\text{and}\quad \sum_{k=2}^{n+1}{}' \frac{|\varrho_k'(0)|}{1-x_k^2} \geqq C_6 n^{1/2},$$

from which it follows that (1. 2. 9) is also best possible, i.e. we can find a polynomial $f_1(x)$ of degree $\leqq 2n+1$ which satisfies (1. 2. 6) and (1. 2. 7) and for a numerical positive C_7

$$f_1'(0) \geqq C_7 (An^{5/2} + Bn^{1/2}).$$

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