On summability of Fourier series

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Introduction

Let Σc_n be a given infinite series and let s_n denote its n-th partial sum. Let $\lambda=\{\lambda_n\}$ be a monotone non-decreasing sequence of integers such that $\lambda_1=1$ and $\lambda_{n+1}-\lambda_n\leqq 1$.

The mean

$$V_n(\lambda)=\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n-1} s_\nu \qquad (n\geqq 1)$$

defines the n-th generalized de la Vallée Poussin mean of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$. The series Σc_n is said to be (V,λ)-summable if $V_n(\lambda)$ converges, and absolutely (V,λ)-summable or, in brief, $|V,\lambda|$-summable if the series

$$\sum_{n=1}^{\infty}|V_{n+1}(\lambda)-V_n(\lambda)|$$

converges.

In the previous papers [7] and [8] we have dealt with (V,λ)-summability of general orthogonal series and $|V,\lambda|$-summability of Fourier series, multiplied by a factor sequence.

The main purpose of the present paper is to unite, in terms of (V,λ)-summation, some classical theorems on the partial sums, the $(\mathscr{C},1)$-means, and the proper de la Vallée Poussin means, of Fourier series. Indeed, it is easy to see that, by suitable choice of $\lambda=\{\lambda_n\}$, the $V_n(\lambda)$ means include the partial sums $(\lambda_n\equiv 1,\ V_{n+1}(\lambda)=s_n)$, the $(\mathscr{C},1)$-means $(\lambda_n=n,\ V_{n+1}(\lambda)=\sigma_n)$ and the proper de la Vallée Poussin means $\left(\lambda_n=\left[\frac{n}{2}\right],^{1)}\ V_{2n}(\lambda)=V_n\right)$, as special cases.

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¹) $[y]$ denotes the integral part of y.

Analogous problems will be investigated for the so-called "strong (V, λ)-summability" (Theorems 5—7).

Furthermore, the method introduced in the present paper shall be used also to give a very simple proof of a theorem on strong $(\mathscr{C}, \alpha, k)$-summability of Fourier series which generalizes ZYGMUND's theorem [13] concerning strong (H, k)-summability (Theorem 8).

Finally we prove some theorems concerning absolute (V, λ)-summability.

Let $f(x)$ be a function integrable in the sense of Lebesgue and periodic with period 2π, and let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

be its Fourier series. $s_n(x)=s_n(f; x)$ and $V_n(\lambda; x)=V_n(f, \lambda; x)$ will denote the n-th partial sum of (1) and the n-th generalized de la Vallée Poussin mean of (1), respectively.

We prove the following theorems:

Theorem 1. *If the function $f(x)$ is bounded, $|f(x)|\leqq M$, then the means $V_n(\lambda; x)$ satisfy the inequality*

$$|V_n(\lambda; x)| \leqq M\left(3+\log\frac{2n-\lambda_n}{\lambda_n}\right). \tag{2}$$

If $\lambda_n=1$ or $\lambda_n=n$, this theorem reduces to classical results of LEBESGUE and FEJÉR, respectively. [2])

We will write as usual $\varphi_x(t) = f(x+2t)+f(x-2t)-2f(x)$ and

$$\Phi_x(h) = \int_0^h |\varphi_x(t)|\,dt.$$

Theorem 2. *If the sequence $\{\lambda_n\}$ tends to infinity and the conditions*

$$\int_{\frac{1}{n}}^{\frac{1}{\lambda_n}} \frac{|\varphi_x(t)|}{t}\,dt = o(1) \qquad (n\to\infty), \tag{3}$$

$$n\Phi_x\left(\frac{1}{n}\right) = o(1) \qquad (n\to\infty) \tag{4}$$

are fulfilled, then the means $V_n(\lambda; x)$ converge to $f(x)$.

[2]) In this and in similar statements it will be understood that the constant factors occurring in the new and the known estimates should not be necessarily the same.

If $n=O(\lambda_n)$, then (3) and (4) are fulfilled at any Lebesgue point of $f(x)$ and consequently almost everywhere, hence this theorem includes the classical Fejér—Lebesgue theorem and the following

Corollary 1. *If the sequence $\{n_n/\lambda\}$ is bounded, then for any integrable $f(x)$ the means $V_n(\lambda; x)$ converge almost everywhere to $f(x)$.*

Let $E_n=E_n(f)$ denote the best approximation to $f(x)$ in the space $\mathscr{C}(0, 2\pi)$ of continuous functions in $(0, 2\pi)$ by a trigonometric polynomial of order not higher than n.

Theorem 3. *If $f(x)$ is continuous, then the estimate*

$$|V_n(\lambda; x)-f(x)| \leqq \left(4+\log\frac{2n-\lambda_n}{\lambda_n}\right) E_{n-\lambda_n}$$

holds true for all x uniformly.

If $\lambda_n\equiv 1$, this theorem gives the well-known result of LEBESGUE [6]. If $\lambda_n=\left[\frac{n}{2}\right]$, then we get the classical theorem of DE LA VALLÉE POUSSIN [12].

Corollary 2. *If there exist two positive numbers, K_1 and K_2, such that $1<K_1\leqq \leqq\frac{n}{\lambda_n}\leqq K_2$, then for any function $f(x)\in \operatorname{Lip}\alpha$ $(0<\alpha\leqq 1)$ the estimate*

$$|V_n(\lambda; x)-f(x)| = O(1/n^\alpha)$$

is valid for all x uniformly.

For a sequence $\{\lambda_n\}$ of general type we have

Theorem 4. *If $f(x)\in\operatorname{Lip}\alpha$, then*

$$|V_n(\lambda; x)-f(x)| = \begin{cases} O\left(\frac{1}{\lambda_n^\alpha}\right) & \text{for} \quad \alpha<1, \\ O\left(\frac{1+\log\lambda_n}{\lambda_n}\right) & \text{for} \quad \alpha=1, \end{cases} \tag{5}$$

for all x uniformly.

This theorem covers BERNSTEIN's results [3] concerning $(\mathscr{C}, 1)$-means.
From Theorem 3 and Lemma 1 we immediately obtain

Corollary 3. *If $n=O(\lambda_n)$, then, for any function $f(x)\in\mathscr{C}(0, 2\pi)$, the means $V_n(\lambda; x)$ converge uniformly.*

We remark that Corollary 2 and 3 can also be deduced from a theorem of EFIMOV [4] (see p. 770).

The strong (V, λ)-summability, i.e., the means of the form

$$T_n(f, \lambda, k; x) \equiv \left\{ \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(x) - f(x)|^k \right\}^{\frac{1}{k}}$$

can also be investigated. We can however prove the strong analogues of the previous theorems only under the restriction $n = O(\lambda_n)$.

Theorem 5. *Suppose that $n = O(\lambda_n)$. Then, for any $f(x) \in \mathscr{C}(0, 2\pi)$ and $k > 0$,*

$$\left\{ \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(x) - f(x)|^k \right\}^{\frac{1}{k}} = O(E_{n-\lambda_n}) \tag{6}$$

holds; and if $|f(x)| \leqq M$, then we have

$$\left\{ \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(x)|^k \right\}^{\frac{1}{k}} = O(M). \tag{7}$$

See also ALEXITS—KRÁLIK [1], Satz 1, and [9], Satz 1.

Theorem 6. *Suppose that $f(x)$ can be differentiated r times and $f^{(r)}(x) \in \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqq 1)$, and that $n = O(\lambda_n)$. Then for any $k > 0$*

$$T_n(f, \lambda, k; x) = \begin{cases} O\left(\dfrac{1}{n^{r+\alpha}}\right) & \text{for} \quad (r+\alpha)k < 1, \\ O\left(\dfrac{1}{n^{r+\alpha}}\left(1 + \log \dfrac{n}{n - \lambda_n + 1}\right)^{\frac{1}{k}}\right) & \text{for} \quad (r+\alpha)k = 1, \end{cases} \tag{8}$$

uniformly. The same estimate is also valid for the conjugate function $\tilde{f}(x)$.

Furthermore, if $(r+\alpha)k = 1$ $(0 < \alpha \leqq 1)$, then there exist functions $f_1(x)$ and $f_2(x)$ such that their r-th derivatives exist and belong to Lip α, *moreover*

$$\overline{\lim_{n\to\infty}}\, T_n(f_1, \lambda, k; 0) \leqq \frac{c}{n^{r+\alpha}} \left(1 + \log \frac{n}{n - \lambda_n + 1}\right)^{\frac{1}{k}} \tag{9}$$

and

$$\overline{\lim_{n\to\infty}}\, T_n(\tilde{f}_2, \lambda, k; 0) \geqq \frac{c}{n^{r+\alpha}} \left(1 + \log \frac{n}{n - \lambda_n + 1}\right)^{\frac{1}{k}}, \tag{10}$$

where $c > 0$ is independent of n.

MARCINKIEWICZ [11] ($k=2$) and ZYGMUND [13] ($k>0$), proved for any integrable function $f(x)$ that

$$\frac{1}{n}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|^k = o(1) \quad \text{a.e.} \tag{11}$$

From this result and Lemma 3 we obtain the following theorems.

Theorem 7. *If $f(x)$ is integrable, then, for any positive k and δ,*

$$\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n}|s_\nu(x)-f(x)|^k = o\left(\left(\frac{n}{\lambda_n}\right)^\delta\right) \quad \text{a.e.} \tag{12}$$

Theorem 8. *If $f(x)$ is an integrable function and k is any positive value, then for any positive α*

$$\frac{1}{A_n^{(\alpha)}}\sum_{\nu+0}^{n}A_{n-\nu}^{(\alpha-1)}|s_\nu(x)-f(x)|^k = o(1) \quad \text{a.e.}, \tag{13}$$

with $A_k^{(\alpha)} = \binom{k+\alpha}{k}$.

It is obvious that similar theorems can be proved on the conjugate function too.

Finally we prove two theorems concerning absolute (V, λ)-summability of Fourier series.

Let $E_n^{(2)} = E_n^{(2)}(f)$ denote the best approximation of $f(x)$ in $L^2(0, 2\pi)$.

Theorem 9. *If*

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\lambda_n}}E_n^{(2)}(f) < \infty, \tag{14}$$

then the Fourier series of $f(x)$ is $|V, \lambda|$-summable a.e.

Theorem 10. *Let $\lambda(x)$ ($x\geqq 1$) be a continuous function, linear between n and $n+1$, furthermore let $\lambda(n)=\sqrt{n\lambda_n}$. If*

$$\int_0^1 \frac{1}{t^2\lambda\left(\frac{1}{t}\right)}\left(\int_0^{2\pi}[f(x+t)-f(x-t)]^2\,dx\right)^{\frac{1}{2}}dt < \infty, \tag{15}$$

then the Fourier series of $f(x)$ is $|V, \lambda|$-summable a.e.

It may be worth while to remark that, under the conditions (14) and (15), the same conclusions are valid for the functions: $|f(x)|$, $\tilde{f}(x)$ and $1/f(x)$ if $|f(x)|\geqq c>0$.

§ 1. Lemmas

Lemma 1. *If* $g(t)\in L\left(0,\frac{\pi}{2}\right)$ *and* $|g(t)|\leqq M$ *in* $(0,\delta)$, $\delta\leqq\frac{\pi}{2}$, *then*

$$I(g,\lambda_n)\equiv\int_0^{\frac{\pi}{2}} g(t)\frac{|\sin\lambda_n t\sin(2n-\lambda_n)t|}{\lambda_n\sin^2 t}\,dt\leqq \tag{1.1}$$
$$\leqq\frac{\pi}{2}M\left(3+\log\frac{2n-\lambda_n}{\lambda_n}\right)+\frac{1}{\lambda_n}J(g,\delta),$$

where $J(g,\delta)$ *is independent of* λ_n *and* $J\left(g,\frac{\pi}{2}\right)=0.$

Proof. *Let* $\alpha_1=\alpha_1(n)=\min\left(\frac{2}{\pi(2n-\lambda_n)},\delta\right)$ and $\alpha_2=\alpha_2(n)=\min\left(\frac{2}{\pi\lambda_n},\delta\right)$. It is evident that

$$I(g,\lambda_n)=\int_0^{\alpha_1}+\int_{\alpha_1}^{\alpha_2}+\int_{\alpha_2}^{\delta}+\int_{\delta}^{\frac{\pi}{2}}\equiv I_1+I_2+I_3+I_4.$$

Each term I_i can easily be estimated by the use of the inequalities $|\sin nt|\leqq n|\sin t|$ and $|\sin t|\geqq\frac{2}{\pi}t$:

$$I_1\leqq\frac{1}{\lambda_n}M\lambda_n(2n-\lambda_n)\int_0^{\alpha_1}dt\leqq\frac{2}{\pi}M,$$

$$I_2\leqq\frac{1}{\lambda_n}\frac{\pi}{2}M\lambda_n\int_{\alpha_1}^{\alpha_2}\frac{1}{t}\,dt\leqq\frac{\pi}{2}M\log\frac{2n-\lambda_n}{\lambda_n},$$

$$I_3\leqq\frac{1}{\lambda_n}\frac{\pi^2}{4}M\int_{\alpha_2}^{\delta}\frac{1}{t^2}\,dt\leqq\frac{\pi^3}{8}M,$$

$$I_4\leqq\frac{1}{\lambda_n}\frac{\pi^2}{4}\int_{\delta}^{\frac{\pi}{2}}\frac{g(t)}{t^2}\,dt\equiv\frac{1}{\lambda_n}J(g,\delta).$$

Summing up, we obtain (1. 1).

Lemma 2. *If* $g(t)\in L(0,2\pi)$ *and* $|g(t)|\leqq M$ *for all* t, *then, for any* $q>0$, *we have*

$$\left\{\frac{1}{m}\sum_{k=1}^{m}|s_k(g;x)|^q\right\}^{\frac{1}{q}}\leqq C_qM. \tag{1.2}$$

Proof. We can assume with no loss in generality that $q \geqq 2$. Indeed, if (1.2) holds for a certain q, then it remains valid for any $q^* > 0$ not greater than q.

Using DIRICHLET's formula for the partial sums we obtain

$$\sum_{k=1}^{m} |s_k|^q \leqq C_1(q) \sum_{k=0}^{m} \left\{ \left(\int_{-\frac{1}{m}}^{\frac{1}{m}} |g(x+t)|\, |D_k(t)|\, dt \right)^q + \right.$$

$$\left. + \left| \int_I g(x+t) D_k(t)\, dt \right|^q \right\} \equiv \sum\nolimits_1 + \sum\nolimits_2,$$

where $D_k(t)$ is DIRICHLET's kernel and I denotes the set $[-\pi, \pi] \setminus \left(-\frac{1}{m}, \frac{1}{m}\right)$.

It is obvious that

$$\Sigma_1 \leqq C_2(q) M^q m$$

and

$$\sum\nolimits_2 \leqq C_3(q) \sum_{k=0}^{m} \left\{ \left| \int_I g(x+t) \operatorname{cotg} \frac{t}{2} \sin kt\, dt \right|^q + \left| \int_I g(x+t) \cos kt\, dt \right|^q \right\} \equiv$$

$$\equiv \sum\nolimits_3 + \sum\nolimits_4.$$

By using the Hausdorff—Young inequality and the notation $r = \dfrac{q}{q-1}$ $(q \geqq 2)$, we get

$$\sum\nolimits_3 \leqq C_4(q) \left(\int_I \frac{|g(x+t)|^r}{|t|^r}\, dt \right)^{\frac{q}{r}} \leqq C_5(q) M^q m$$

and

$$\sum\nolimits_4 \leqq C_6(q) \left(\int_I |g(x+t)|^r\, dt \right)^{\frac{q}{r}} \leqq C_7(q) M^q.$$

Collecting our estimates, we obtain (1.2).

Lemma 3. *Let $\{c_n\}$ be a given sequence. If for any positive β*

$$\frac{1}{n} \sum_{\nu=0}^{n} |c_\nu|^\beta = o(1), \tag{1.3}$$

then, for any triangular matrix $\|\alpha_{n\nu}\|$ and for any positive γ and $p > 1$, we have the estimate

$$\sum_{\nu=0}^{n} \alpha_{n\nu} |c_\nu|^\gamma = o\left(n^{1-\frac{1}{p}} \left\{ \sum_{\nu=0}^{n} |\alpha_{n\nu}|^p \right\}^{\frac{1}{p}} \right). \tag{1.4}$$

Proof. By HÖLDER's inequality, we get

$$\sum_{\nu=0}^{n} \alpha_{n\nu}|c_\nu|^\gamma \leq \left\{\sum_{\nu=0}^{n} |c_\nu|^{\frac{\gamma p}{p-1}}\right\}^{\frac{p-1}{p}} \left\{\sum_{\nu=0}^{n} |\alpha_{n\nu}|^p\right\}^{\frac{1}{p}} = o\left(n^{1-\frac{1}{p}}\right)\left\{\sum_{\nu=0}^{n} |\alpha_{n\nu}|^p\right\}^{\frac{1}{p}},$$

which is the required statement.

§ 2. Proof of the theorems

Ad Theorem 1. A standard computation gives that

$$V_n(\lambda; x) = \frac{1}{\lambda_n \pi} \int_0^{\frac{\pi}{2}} [f(x+2t)+f(x-2t)] \frac{\sin \lambda_n t \sin (2n-\lambda_n)t}{\sin^2 t} dt. \tag{2. 1}$$

Applying Lemma 1 with $g(t) = f(x+2t)-f(x-2t)$ and $\delta = \frac{2}{\pi}$, we get the inequality (2).

Ad Theorem 2. Since the (V, λ)-means of a constant function $f(x) \equiv c$ equal to c, from (2. 1) it follows that

$$V_n(\lambda; x)-f(x) = \frac{1}{\lambda_n \pi} \int_0^{\frac{\pi}{2}} \varphi_x(t) \frac{\sin \lambda_n t \sin (2n-\lambda_n)t}{\sin^2 t} dt. \tag{2. 2}$$

Divide the integral into three parts:

$$\int_0^{\frac{\pi}{2}} = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{1}{\lambda_n}} + \int_{\frac{1}{\lambda_n}}^{\frac{\pi}{2}} \equiv I_1+I_2+I_3.$$

These integrals can easily be estimated by standard methods:

$$I_1 \leq O(1) \int_0^{\frac{1}{n}} |\varphi_x(t)| \lambda_n n \, dt = O\left(\lambda_n n \Phi_x\left(\frac{1}{n}\right)\right),$$

$$I_2 \leq O(1) \int_{\frac{1}{n}}^{\frac{1}{\lambda_n}} |\varphi_x(t)| \frac{\lambda_n t}{t^2} dt = O\left(\lambda_n \int_{\frac{1}{n}}^{\frac{1}{\lambda_n}} \frac{|\varphi_x(t)|}{t} dt\right),$$

$$I_3 \leq O(1) \int_{\frac{1}{\lambda_n}}^{\frac{\pi}{2}} \frac{|\varphi_x(t)|}{t^2} dt \leq O(1)[\Phi_x(t)t^{-2}]_{\frac{1}{\lambda_n}}^{\frac{\pi}{2}} + O(1) \int_{\frac{1}{\lambda_n}}^{\frac{\pi}{2}} \frac{\Phi_x(t)}{t^3} dt \leq$$

$$\leq O(1)+O\left(\lambda_n^2 \Phi_x\left(\frac{1}{\lambda_n}\right)\right)+o(\lambda_n)+O(1) \leq O\left(1+\lambda_n^2 \Phi_x\left(\frac{1}{\lambda_n}\right)\right)+o(\lambda_n).$$

From these estimates we get that

$$V_n(\lambda; x)-f(x) = O\left(n\Phi_x\left(\frac{1}{n}\right)+\int_{\frac{1}{n}}^{\frac{1}{\lambda_n}} \frac{|\varphi_x(t)|}{t}\,dt+\lambda_n\Phi_x\left(\frac{1}{\lambda_n}\right)\right)+o(1),$$

which gives the conclusion of Theorem 2 by (3) and (4).

Ad Theorem 3. Let $T_m^*(x)$ denote the trigonometric polynomial of best approximation of order not higher than m. From the definitions of $s_n(f; x)$ and $V_n(f, \lambda; x)$ it is clear that if $n-\lambda_n \geqq m$, then

$$V_n(f-T_m^*, \lambda; x) = V_n(f, \lambda; x)-T_m^*(x).$$

Hence we have

$$\begin{aligned}|V_n(f, \lambda; x)-f(x)| &\leqq |V_n(f, \lambda; x)-T_{n-\lambda_n}^*(x)|+|T_{n-\lambda_n}^*(x)-f(x)| \leqq \\ &\leqq |V_n(f-T_{n-\lambda_n}^*, \lambda; x)|+E_{n-\lambda_n}.\end{aligned} \tag{2. 3}$$

According to (2. 1) we get

$$|V_n(f-T_{n-\lambda_n}^*, \lambda; x)| \leqq \frac{1}{\lambda_n\pi}\int_0^{\frac{\pi}{2}} 2E_{n-\lambda_n}\frac{|\sin\lambda_n t\sin(2n-\lambda_n)t|}{\sin^2 t}\,dt.$$

Using Lemma 1 with $g(t)\equiv 2E_{n-\lambda_n}$ and $\delta=\frac{\pi}{2}$, we obtain

$$|V_n(f-T_{n-\lambda_n}^*, \lambda; x)| \leqq E_{n-\lambda_n}\left(3+\log\frac{2n-\lambda_n}{\lambda_n}\right). \tag{2. 4}$$

The statement of Theorem 3 follows from (2. 3) and (2. 4).

Ad Theorem 4. *If* $f(x)\in\mathrm{Lip}\,\alpha$, we obtain, using (2. 2),

$$|V_n(\lambda; x)-f(x)| \leqq O(1)\frac{1}{\lambda_n}\int_0^{\frac{\pi}{2}} t^\alpha\frac{|\sin\lambda_n t\sin(2n-\lambda_n)t|}{t^2}\,dt.$$

Let us split the integral into three parts:

$$\int_0^{\frac{\pi}{2}} = \int_0^{\frac{1}{2n-\lambda_n}} + \int_{\frac{1}{2n-\lambda_n}}^{\frac{1}{\lambda_n}} + \int_{\frac{1}{\lambda_n}}^{\frac{\pi}{2}} \equiv I_1+I_2+I_3.$$

I_1 and I_2 can easily be estimated for any $\alpha \leqq 1$ as follows.

$$\frac{1}{\lambda_n} I_1 \leqq \frac{1}{\lambda_n} \int_0^{\frac{1}{2n-\lambda_n}} t^\alpha \lambda_n (2n-\lambda_n)\, dt \leqq \frac{1}{(2n-\lambda_n)^\alpha} \leqq \frac{1}{\lambda_n^\alpha},$$

$$\frac{1}{\lambda_n} I_2 \leqq \frac{1}{\lambda_n} \int_{\frac{1}{2n-\lambda_n}}^{\frac{1}{\lambda_n}} t^{\alpha-1} \lambda_n\, dt \leqq \frac{1}{\alpha} \frac{1}{\lambda_n^\alpha}.$$

The estimate of I_3 differs according to whether $\alpha<1$ or $\alpha=1$:

$$\frac{1}{\lambda_n} I_3 \leqq \frac{1}{\lambda_n} \int_{\frac{1}{\lambda n}}^{\frac{\pi}{2}} t^{\alpha-2}\, dt \leqq \begin{cases} \dfrac{1}{1-\alpha} \dfrac{1}{\lambda_n^\alpha}, & \text{if} \quad \alpha < 1 \\[2ex] \dfrac{1+\log \lambda_n}{\lambda_n}, & \text{if} \quad \alpha = 1. \end{cases}$$

Collecting our estimates, we obtain (5).

Ad Theorem 5. Using the notations introduced in the proof of Theorem 3, it is obvious that if $\nu \geqq m$, then

$$s_\nu(f-T_m^*; x) = s_\nu(f; x) - T_m^*(x).$$

From this it follows that

$$\left\{\frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(x)-f(x)|^k\right\}^{\frac{1}{k}} \leqq \left\{\frac{2^k}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} \left(|s_\nu(f-T_{n-\lambda_n}^*; x)|^k + \right.\right. \tag{2.5}$$

$$\left.\left. + |T_{n-\lambda_n}^*(x)-f(x)|^k\right)\right\}^{\frac{1}{k}} \leqq 2^{1+\frac{1}{k}} \left(\left\{\frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(f-T_{n-\lambda_n}^*; x)|^k\right\}^{\frac{1}{k}} + E_{n-\lambda_n}\right).$$

Applying Lemma 2 (with $g(t)=f(t)-T_{n-\lambda_n}^*(t)$ and $q=k$), we get the required estimate (6) by (2. 5).

The statement can immediately be derived from Lemma 2.

The proof of Theorem 5 is thus completed.

Ad Theorem 6. The assumption $f^{(r)}(x) \in \mathrm{Lip}\,\alpha$ implies that $E_n(f)=O\left(\frac{1}{n^{r+\alpha}}\right)$ and $E_n(\tilde{f})=O\left(\frac{1}{n^{r+\alpha}}\right)$ (see [14], (13. 14) Theorem, p. 117., and (13. 29) Theorem, p. 121). From these and (6) we get, with $\lambda_n=\left[\frac{n}{2}\right]$ and $n=2m$, that

$$h_m(f, k; x) \equiv \left\{\frac{1}{m} \sum_{\nu=m}^{2m-1} |s_\nu(x)-f(x)|^k\right\}^{\frac{1}{k}} = O\left(\frac{1}{m^{r+\alpha}}\right) \tag{2.6}$$

and

$$h_m(\tilde{f}, k; x) \equiv \left\{\frac{1}{m}\sum_{\nu=m}^{2m-1} |\tilde{s}_\nu(x)-\tilde{f}(x)|^k\right\}^{\frac{1}{k}} = O\left(\frac{1}{m^{r+\alpha}}\right),$$

where $\tilde{s}_\nu(x)=s_\nu(\tilde{f}; x)$.

Suppose that $2^{m_1} \leqq n-\lambda_n+1 < 2^{m_1+1}$ and $2^{m_2} < n \leqq 2^{m_2+1}$. Then, by (2. 6), we have

$$\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n-1} |s_\nu(x)-f(x)|^k \leqq \frac{1}{\lambda_n}\sum_{m=m_1}^{m_2}\sum_{\nu=2^m}^{2^{m+1}-1} |s_\nu(x)-f(x)|^k \leqq$$

$$\leqq \frac{O(1)}{\lambda_n}\sum_{m=m_1}^{m_2} 2^{m(1-k(r+\alpha))} \equiv \sum\nolimits_1 .$$

If $k(r+\alpha)<\alpha$, then, by $n=O(\lambda_n)$ and $2^{m_2}<n$,

$$\sum\nolimits_1 \leqq O(1)\frac{1}{\lambda_n} 2^{m_2(1-k(r+\alpha))} = O\left(\frac{1}{n^{k(r+\alpha)}}\right);$$

and if $k(r+1)=1$, then

$$\sum\nolimits_1 \leqq O(1)\frac{1}{\lambda_n}(m_2-m_1) = O\left(\frac{1}{n}\left(1+\log\frac{n}{n-\lambda_n+1}\right)\right).$$

From these estimates the statement (8) obviously follows.

The statement for the conjugate function can similarly be verified.

In order to prove the statements (9) and (10) let us define the following functions. If r is an even integer, let

$$f_1(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^{1+r+\alpha}}$$

and

$$f_2(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n\alpha}}\sum_{l=2^{n-1}+1}^{2^n}\left(\frac{\sin(5\cdot 2^n-l)x}{(5\cdot 2^n-l)^r l}-\frac{\sin(5\cdot 2^n+l)x}{(5\cdot 2^n+l)^r l}\right);$$

and if r is an odd integer, let

$$f_1(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n\alpha}}\sum_{l=2^{n-1}+1}^{2n}\left(\frac{\cos(5\cdot 2^n-l)x}{(5\cdot 2^n-l)^r l}-\frac{\cos(5\cdot 2^n+l)x}{(5\cdot 2^n+l)^r l}\right)$$

and

$$f_2(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^{1+r+\alpha}}.$$

It is proved that the r-th derivatives of these functions belong to Lip α (see [9], Satz IV). We shall now verify the inequalities (9) and (10) only for even r since the other case would be an analogous computation.

On account of $(r+\alpha)k=1$ and $n=O(\lambda_n)$, we have

$$T_n(f_1,\lambda,k;0)=\left\{\frac{1}{\lambda_n}\sum_{v=n-\lambda_n}^{n-1}\left(\sum_{m=v+1}^{\infty}\frac{1}{m^{r+1+\alpha}}\right)^k\right\}^{\frac{1}{k}}\geqq$$

$$\geqq C_1(k)\left\{\frac{1}{\lambda_n}\sum_{v=n-\lambda_n}^{n-1}\frac{1}{v}\right\}^{\frac{1}{k}}\geqq C_2(k)\left\{\frac{1}{n}\log\frac{n}{n-\lambda_n+1}\right\}^{\frac{1}{k}}\geqq$$

$$\geqq C_2(k)\left\{\frac{1}{n}\left(1+\log\frac{n}{n-\lambda_n+1}\right)\right\}^{\frac{1}{k}}\geqq C_4(k)n^{-r-\alpha}\left(1+\log\frac{n}{n-\lambda_n+1}\right)^{\frac{1}{k}},$$

which gives (9).

The proof of (10) is more complicated. We will first prove (10) in case that the sequence $\left\{\frac{n}{n-\lambda_n}\right\}$ is bounded. Let $n=12\cdot 2^m$ and let $\alpha_1=\max(n-\lambda_n, 22\cdot 2^{m-1})$, $\alpha_2=\max(\alpha_1, 23\cdot 2^{m-1})$ and $\alpha_3=\max\left(\alpha_2, n-\left[\frac{\lambda_n+1}{2}\right]\right)$. Then

$$T_n(\tilde f_2,\lambda,k;0)=\left\{\frac{1}{\lambda_n}\sum_{v=n-\lambda_n}^{n-1}|\tilde s_v(0)-\tilde f(0)|^k\right\}^{\frac{1}{k}}\geqq$$

$$\geqq\left\{\frac{1}{\lambda_n}\left(\sum_{v=\alpha_1}^{\alpha_2-1}+\sum_{v=\alpha_2}^{\alpha_3}\right)\left|\frac{1}{n^\alpha}\sum_{l=v-10\cdot 2^m+1}^{2^{m+1}}\frac{1}{n^r l}\right|^k\right\}^{\frac{1}{k}}.$$

It is clear by $n=O(\lambda_n)$, that

$$\sum_{v=\alpha_1}^{\alpha_2-1}\left|\frac{1}{n^\alpha}\sum_{l=v-10\cdot 2^m+1}^{2^{m+1}}\frac{1}{n^r l}\right|^k\geqq(\alpha_2-\alpha_1)\left|\frac{1}{n^\alpha}\sum_{l=\alpha_2-10\cdot 2^m}^{2^{m+1}}\frac{1}{n^r l}\right|^k\geqq$$

$$\geqq(\alpha_2-\alpha_1)\left|\frac{1}{n^{r+1+\alpha}}\min(2^{m-1},\lambda_n)\right|^k\geqq C_1(k)(\alpha_2-\alpha_1)\frac{1}{n^{(r+\alpha)k}}.$$

Similarly,

$$\sum_{v=\alpha_2}^{\alpha_3}\left|\frac{1}{n^\alpha}\sum_{l=v-10\cdot 2^m+1}^{2^{m+1}}\frac{1}{n^r l}\right|^k\geqq(\alpha_3-\alpha_2)\left|\frac{1}{n^\alpha}\sum_{l=\alpha_3-10\cdot 2^m}^{2^{m+1}}\frac{1}{n^r l}\right|^k\geqq$$

$$\geqq(\alpha_3-\alpha_2)\left|\frac{1}{n^{r+\alpha+1}}\min\left(2^{m-1},\frac{\lambda_n}{n}\right)\right|^k\geqq C_2(k)(\alpha_3-\alpha_2)\frac{1}{n^{(r+\alpha)k}}.$$

Hence we get

$$T_n(\tilde f_2,\lambda,k;0)\geqq C_3(k)\left\{(\alpha_3-\alpha_1)\frac{1}{\lambda_n}\frac{1}{n^{(r+\alpha)k}}\right\}^{\frac{1}{k}}\geqq C_4(k)\frac{1}{n^{r+\alpha}}$$

which proves (10) under the assumption that $\left\{\frac{n}{n-\lambda_n}\right\}$ is bounded.

If $\frac{n}{n-\lambda_n}$ tends to infinity, then the proof is simpler. Suppose that $4\cdot 2^m < n \leqq \leqq 4\cdot 2^{m+1}$ and $4\cdot 2^\mu \leqq n-\lambda_n+4 < 4\cdot 2^{\mu+1}$; and that $m > \mu+2$. Then we have

$$(2.7)\quad \begin{aligned} T_n(\tilde{f}_2, \lambda, k; 0) &\geqq \left\{\frac{1}{\lambda_n}\sum_{p=\mu+1}^{m-1}\sum_{\nu=4\cdot 2^p+1}^{4\cdot 2^{p+1}} |\tilde{s}_\nu(0)-\tilde{f}(0)|^k\right\}^{\frac{1}{k}} \geqq \\ &\geqq \left\{\frac{1}{\lambda_n}\sum_{p=\mu+1}^{m-1}\sum_{\nu=11\cdot 2^{p-1}}^{12\cdot 2^{p-1}} |\tilde{s}_\nu(0)-\tilde{f}(0)|^k\right\}^{\frac{1}{k}} \equiv \left\{\frac{1}{\lambda_n}\sum_{p=\mu+1}^{m-1} I_p\right\}^{\frac{1}{k}}. \end{aligned}$$

I_p can easily be estimated as follows

$$\begin{aligned} I_p &\geqq \sum_{\nu=11\cdot 2^{p-1}}^{23\cdot 2^{p-2}} \left(\frac{1}{2^{p\alpha}}\sum_{l=\nu-10\cdot 2^{p-1}+1}^{2p} \frac{1}{6^r\cdot 2^{pr}l}\right)^k \geqq \\ &\geqq \sum_{\nu=11\cdot 2^{p-1}}^{23\cdot 2^{p-2}} \left(\frac{1}{2^{p\alpha}}\sum_{l=23\cdot 2^{p-2}-10\cdot 2^{p-1}+1}^{2^p} \frac{1}{6^r 2^{pr}l}\right)^k \geqq \\ &\geqq C_1(r,k)2^{p-2}\frac{1}{2^{p(r+\alpha)k}} = C_2(r,k) \equiv C_2 > 0. \end{aligned}$$

From this and (2.7) we get

$$T_n(\tilde{f}_2, \lambda, k; 0) \geqq C_2^{\frac{1}{k}}\left(\frac{1}{\lambda_n}(m-\mu-2)\right)^{\frac{1}{k}} = C_2^{\frac{1}{k}}\left\{\frac{1}{n}\left(\frac{1}{\log 2}\log\frac{n}{n-\lambda_n+1}-6\right)\right\}^{\frac{1}{k}}.$$

Since $(r+\alpha)k=1$ and $\frac{n}{n-\lambda_n+1}\to\infty$, it is easy to see that, for n large enough,

$$T_n(\tilde{f}_2, \lambda, k; 0) \geqq \frac{c}{n^{r+\alpha}}\left(1+\log\frac{n}{n-\lambda_n+1}\right)^{\frac{1}{k}}$$

where $c>0$ is independent of n.

We have completed our proof.

Ad Theorem 7. To prove (12), by (11) and (1.4), it is sufficient to choose $p(>1)$ so close to 1 that $1-\frac{1}{p}\leqq\delta$, which is possible.

Ad Theorem 8. In order to prove (13), it is sufficient to show that, by suitable choice of p occurring in (1.4), the estimate

$$(2.8)\quad n^{1-\frac{1}{p}}\left\{\sum_{\nu=0}^{n}(A_{n-\nu}^{(\alpha-1)})^p\right\}^{\frac{1}{p}} = O(A_n^\alpha)$$

holds true. Let us choose p such that $(\alpha-1)p>-1$. Then we have

$$\sum_{\nu=0}^{n}(A_{n-\nu}^{(\alpha-1)})^p=\left(\sum_{\nu=0}^{\left[\frac{n}{2}\right]}+\sum_{\nu=\left[\frac{n}{2}\right]+1}^{n}\right)(A_{n-\nu}^{(\alpha-1)})^p\leqq$$

$$\leqq O(1)\sum_{\nu=0}^{\left[\frac{n}{2}\right]}n^{(\alpha-1)p}+O(1)\sum_{\nu=1}^{\left[\frac{n}{2}\right]}\nu^{(\alpha-1)p}=O(n^{1+(\alpha-1)p}).$$

From this we obtain

$$n^{1-\frac{1}{p}}\left\{\sum_{\nu=0}^{n}(A_{n-\nu}^{(\alpha-1)})^p\right\}^{\frac{1}{p}}=O(n^\alpha)$$

which is equivalent to (2. 8) as was required to be proved.

Ad Theorems 9 and 10. In [10] it is proved that if $\bar\lambda(x)$ is a positive, monotonic function with the property

$$\sum_{n=m}^{\infty}\frac{1}{n\bar\lambda(n)}\leqq K\frac{1}{\bar\lambda(m)},\tag{2. 9}$$

then the conditions

$$\sum_{n=1}^{\infty}\frac{1}{\bar\lambda(n)}E_n^{(2)}(f)<\infty,$$

$$\int_0^1\frac{1}{t^2\bar\lambda\left(\frac{1}{t}\right)}\left(\int_0^{2\pi}[f(x+t)-f(x-t)]^2\,dx\right)^{\frac{1}{2}}dt<\infty$$

are equivalent.

The function $\lambda(x)$ occurring in Theorem 9 satisfies (2. 9), viz.

$$\sum_{n=m}^{\infty}\frac{1}{n\sqrt{n\lambda_n}}\leqq\frac{1}{\sqrt{\lambda_m}}\sum_{n=m}^{\infty}\frac{1}{n^{3/2}}\leqq K\frac{1}{\sqrt{m\lambda_m}};$$

(14) and (15) are therefore equivalent.

By virtue of the equivalence, it is sufficient to prove that condition (14) implies the $|V,\lambda|$-summability of (1).

We proved in [7] (Satz VIII) that any orthogonal series $\sum c_n\varphi_n(x)$ is $|V,\lambda|$-summable a.e. under the condition

$$\sum_{m=0}^{\infty}\left(\sum_{n=\mu_m+1}^{\mu_{m+1}}c_n^2\right)^{\frac{1}{2}}<\infty,$$

where $\mu_0=1$ and $\mu_m=\sum_{k=0}^{m-1}\lambda_{\mu_k}$. Therefore, it only remains to prove that the inequality

$$\sum_{m=0}^{\infty}C_m\equiv\sum_{m=0}^{\infty}\left\{\sum_{n=\mu_m+1}^{\mu_{m+1}}(a_n^2+b_n^2)\right\}^{\frac{1}{2}}<\infty\tag{2. 10}$$

follows from condition (14). Since $\sum_{k=n+1}^{\infty}(a_k^2+b_k^2)=(E_n^{(2)}(f))^2$, we obtain by a simple computation that

$$(2.11)\qquad \sum_{n=2}^{\infty} C_n = \sum_{m=1}^{\infty}\sum_{n=2^m}^{2^{m+1}-1} C_n \leqq \sum_{m=1}^{\infty} 2^{\frac{m}{2}}\left\{\sum_{n=2^m}^{2^{m+1}-1} C_n^2\right\}^{\frac{1}{2}} \leqq \sum_{m=1}^{\infty} 2^{\frac{m}{2}} E_{\mu_{2^m}}^{(2)}.$$

If we can show that

$$(2.12)\qquad \sum_{n=\mu_{2^{m-1}}+1}^{\mu_{2^m}} \frac{1}{\sqrt{n\lambda_n}} \geqq \frac{1}{4} 2^{\frac{m}{2}},$$

then, by (2. 11) and (2. 12), we have

$$\sum_{n=2}^{\infty} C_n \leqq 4\sum_{m=1}^{\infty}\sum_{n=\mu_{2^{m-1}}+1}^{\mu_{2^m}} \frac{1}{\sqrt{n\lambda_n}} E_{\mu_{2^m}}^{(2)} \leqq 4\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\lambda_n}} E_n^{(2)},$$

i.e. the condition (14) implies (2. 10) indeed as we stated.

It remains to prove (2. 12). Using the definition of μ_n, we get

$$\sum_{n=\mu_{2^{m-1}}+1}^{\mu_{2^m}} \frac{1}{\sqrt{n\lambda_n}} \geqq \sum_{k=2^{m-1}}^{2^m-1}\sum_{n=\mu_k+1}^{\mu_{k+1}} \frac{1}{\sqrt{n\lambda_n}} \geqq \sum_{k=2^{m-1}}^{2^m-1} \lambda_{\mu_k}\frac{1}{\sqrt{\mu_{k+1}\lambda_{\mu_{k+1}}}} \geqq$$

$$\geqq \sum_{k=2^{m-1}}^{2^m-1} \lambda_{\mu_k}\frac{1}{2\sqrt{\mu_k\lambda_{\mu_k}}} \geqq \frac{1}{2}\sum_{k=2^{m-1}}^{2^m-1}\left(\frac{\lambda_{\mu_k}}{\sum_{i=0}^{k-1}\lambda_{\mu_i}}\right)^{\frac{1}{2}} \geqq \frac{1}{2}\sum_{k=2^{m-1}}^{2^m-1}\frac{1}{\sqrt{k}} \geqq \frac{1}{4}2^{\frac{m}{2}},$$

and this completes the proof.

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