On a theorem of L. Takács

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L. TAKÁCS [1] proved the following generalization of the classical ballot theorem in probability theory:

Let $\Phi(u)$ be a nondecreasing function on the interval $I=[0, t]$, for which $\Phi'(u)=0$ almost everywhere and $\Phi(0)=0$. Set $\Phi(t+u)=\Phi(t)+\Phi(u)$ for $0 \leqq u \leqq t$. Define the function δ on I as follows; $\delta(u)=1$ if $\Phi(v)-\Phi(u) \leqq v-u$ for every v such that $u \leqq v \leqq u+t$, and $\delta(u)=0$ otherwise. Then

$$\int_0^t \delta(u)\,du = t-\Phi(t) \qquad \text{whenever } \Phi(t) \leqq t.$$

In the following we shall give a new proof of this theorem or rather of a somewhat more general theorem, which we formulate in a measure theoretic form:

Theorem. *Let m and μ be non-negative complete regular measures on a circle C such that m is absolutely continuous and μ is singular with respect to the Lebesgue measure on C, and $m(C) \geqq \mu(C)$. Denote by $\Delta[m, \mu]$ the set of the points $x \in C$ for which*

$$m[x, y) \geqq \mu[x, y)$$

holds for all $y \in C$, where $[x, y)$ denotes the semi-closed arc from x to y in the positive direction. Then $\Delta[m, \mu]$ is m-measurable (as well as Lebesgue measurable) and we have

$$m(\Delta[m, \mu]) = m(C)-\mu(C). \tag{1}$$

In the particular case that m is the Lebesgue measure, our theorem is obviously equivalent to TAKÁCS's theorem.

The proof of the theorem will be done in several steps.

1. First suppose that μ is concentrated into finitely many points $x_1, x_2, \cdots, x_n$ and that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the corresponding measures. Then we prove the theorem by induction on n. The case $n=1$ is trivial. Indeed, if we associate with x_1 the minimal arc $(\xi_1, x_1]$ of measure α_1 (m being continuous with $m(C) \geqq \mu(C)$, this can be done), the complementary set of $(\xi_1, x_1]$ coincides with $\Delta[m, \mu]$. Suppose the theorem is

true for n and then we prove it for $n+1$. Let μ be concentrated in the points $x_1, x_2, \cdots, x_{n+1}$ with the corresponding measures $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$. We associate with every point x_i $(i=1, 2, \cdots, n+1)$ the minimal arc $(\xi_i, x_i]$ of m-measure α_i. If these arcs are mutually non-overlapping then the complementary set of the union of these arcs coincides with $\Delta[m, \mu]$, which proves our statement. In case that there exists an x_k contained in some $(\xi_i, x_i]$, consider the atomic measure μ' that we obtain from μ by shifting the measure α_k from the point x_k to the point x_i so that x_k will have measure 0 and x_i will have measure $\alpha_i+\alpha_k$. This measure μ' has only n atoms and we have $\Delta[m, \mu]=\Delta[m, \mu']$. Using our induction hypothesis, our theorem is proved.

2. Let μ be an arbitrary atomic measure, its atoms being at the points $x_1, x_2, \cdots$, with the measures $\alpha_1, \alpha_2, \cdots$, respectively. Denote by μ_n $(n=1, 2, \cdots)$ the atomic measure whose atoms are at $x_1, x_2, \cdots, x_n$ with the corresponding measures $\alpha_1, \alpha_2, \cdots, \alpha_n$. Our statement easily follows from the fact $\Delta[m, \mu]=\bigcap_{n=1}^{\infty} \Delta[m, \mu_n]$.

Let namely $x\in\Delta[m, \mu]$, then a fortiori we have $x\in\Delta[m, \mu_n]$ for every n, which gives that $\Delta[m, \mu]\subseteq\bigcap_{n=1}^{\infty}\Delta[m, \mu_n]$. Conversely, suppose that $x\in\bigcap_{n=1}^{\infty}\Delta[m, \mu_n]$. Then for any n and $y\in C$ we have $m[x, y)\geqq\mu_n[x, y)$, from which we get $m[x, y)\geqq \lim_{n\to\infty}\mu_n[x, y)=\mu[x, y)$, i.e. $x\in\Delta[m, \mu]$. Summarizing, we obtain, that $\Delta[m, \mu]=\bigcap_{n=1}^{\infty}\Delta[m, \mu_n]$, which was to be proved.

3. Let μ be an arbitrary singular measure and denote by Z a set of (Lebesgue) measure 0 such that μ vanishes on every measurable set contained in the complementary set of Z. Choose to Z countable many covering systems $\{I_i^{(n)}\}_{i=1}^{\infty}$ $(n=1, 2, \cdots)$ of disjoint open arcs such that

$$\bigcup_{i=1}^{\infty} I_i^{(n)} \supset \bigcup_{i=1}^{\infty} I_i^{(n+1)} \qquad \text{and} \qquad \sum_{i=1}^{\infty} m(I_i^{(n)}) < \frac{1}{2^n} \qquad (n = 1, 2, \ldots).$$

To the n-th covering system we associate the atomic measure ν_n whose atoms are at the starting points of the arcs $I_i^{(n)}$ $(i=1, 2, \cdots)$, each with measure $\mu(I_i^{(n)})$. Set

$$\Delta_n = \Delta[m, \nu_n] \cup \left(\bigcup_{i=1}^{\infty} I_i^{(n)}\right).$$

Then we have

$$\Delta_1 \supset \Delta_2 \supset \cdots \qquad \text{and} \qquad \Delta[m, \mu] \subset \bigcap_{n=1}^{\infty} \Delta_n.$$

In an analogous manner we construct a second system of atomic measures by associating with the n-th covering system an atomic measure ν_n' whose atoms are at the terminal points of the arcs $I_i^{(n)}$ $(i=1, 2, \cdots)$, each with measure $\mu(I_i^{(n)})$. Set

$$\Delta_n' = \Delta[m, v_n'] - \left(\bigcup_{i=1}^{\infty} I_i^{(n)}\right);$$

it can be easily seen that

$$\Delta_1' \subset \Delta_2' \subset \cdots \qquad \text{and} \qquad \Delta[m, \mu] \supset \bigcup_{n=1}^{\infty} \Delta_n'.$$

Summarizing, we obtain $\bigcup_{n=1}^{\infty} \Delta_n' \subset \Delta[m, \mu] \subset \bigcap_{n=1}^{\infty} \Delta_n$. Since

$$\lim_{n\to\infty} m(\Delta_n) = \lim_{n\to\infty} m(\Delta_n') = m(C) - \mu(C),$$

we get that $\Delta[m, \mu]$ is m-measurable and $m(\Delta[m, \mu]) = m(C) - \mu(C)$, i. e. (1).

Remark. The theorem remains true if m is only continuous, but μ is singular *with respect to m*. The proof is essentially the same.

Reference

[1] L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes* (New York—London—Sydney, 1967).

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