On mixing sequences of σ-algebras

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1. Let $\{\Omega, \mathscr{A}, P\}$ be a probability space, i.e. let $\mathscr{A}$ be a σ-algebra of some subsets of the basic set Ω and for $A \in \mathscr{A}$ let $P(A)$ denote the probability measure of A. Random variables are defined as functions on Ω, which are measurable with respect to the σ-algebra $\mathscr{A}$. The elements of $\mathscr{A}$ will be called events. $\bar{A}$ denotes the event consisting in the non-occurrence of the event A. The elements of Ω will be denoted by ω. If A is an event then χ_A denotes the indicator of A, i.e. $\chi_A = 1$, if $\omega \in A$ and $\chi_A = 0$, if $\omega \notin A$. Let $\mathscr{A}' \subset \mathscr{A}$ be another σ-algebra of some subsets of Ω. We denote by $M(\xi|\mathscr{A}')$ the conditional expectation of the random variable ξ, i.e. such a random variable, which is measurable with respect to $\mathscr{A}'$ and for which

$$\int_A M(\xi|\mathscr{A}')\, dP = \int_A \xi\, dP$$

holds, if $A \in \mathscr{A}'$. If, especially, $\xi = \chi_A$ then $M(\chi_A|\mathscr{A}')$ $(A \in \mathscr{A})$ will be called the conditional probability of the event A with respect to $\mathscr{A}'$ and will be denoted by $P(A|\mathscr{A}')$. The simplest properties of the conditional expectation will be used in this paper.

A. RÉNYI ([1]) calls the sequence $\{B_n\}$ of random events *mixing with density d* if for any fixed random event E

$$\lim_{n \to +\infty} P(B_n E) = dP(E) \tag{1}$$

holds, where d is a fixed number, $0 < d < 1$. It follows that $\lim_{n \to +\infty} P(B_n) = d$. For mixing sequences $\{B_n\}$ with density d the following limit relation also holds: for every fixed random event E

$$\lim_{n \to +\infty} P(E|B_n) = P(E). \tag{1'}$$

A sequence $\{B_n\}$, satisfying the relation (1′), is *mixing* in the sense of the definition of A. RÉNYI if the limit $\lim_{n \to +\infty} P(B_n) = d$ exists and $0 < d < 1$.

It is obvious that, if the sequence $\{B_n\}$ is mixing with density d, then the sequence $\{\bar{B}_n\}$ is also mixing with density $1-d$. Thus we have for every fixed E

$$\lim_{n \to +\infty} P(E|\bar{B}_n) = P(E). \tag{1''}$$

The facts expressed by (1′) and (1″) can be unified in the following manner: let $\mathscr{F}_n$ denote the class $\{\Omega, B_n, \bar{B}_n, O\}$ of sets, where O is the empty set and Ω is the basic set. It is easily seen that $\mathscr{F}_n$ is a σ-algebra. For every fixed event E we have with probability 1

$$P(E|\mathscr{F}_n) = P(E|B_n)\chi_{B_n} + P(E|\bar{B}_n)(1-\chi_{B_n}),$$

where $P(E|\mathscr{F}_n)$ denotes the conditional probability of E with respect to the σ-algebra $\mathscr{F}_n$, χ_{B_n} and $1-\chi_{B_n}$ are the indicators of the events B_n and $\bar{B}_n$, respectively. From the fact that $\{B_n\}$ is a mixing sequence with density d, it follows that we have with probability 1

(2) $$\lim_{n\to+\infty} P(E|\mathscr{F}_n) = P(E)$$

and so this limit relation holds in probability measure, too.

However, relation (2) implies the mixing property (1) of the sequences $\{B_n\}$ and $\{\bar{B}_n\}$ only if $\lim_{n\to+\infty} P(B_n) = d$ exists and $0<d<1$.

Relation (2) suggests a more general formulation of the notion of mixing sequences of events.

Let $\{\mathscr{G}_n\}$ $(n=1, 2, \cdots; \mathscr{G}_n \subset \mathscr{A})$ be a sequence of σ-algebras.

Definition. The sequence $\{\mathscr{G}_n\}$ of σ-algebras is called mixing if for every fixed event E the sequence

(3) $$P(E|\mathscr{G}_n) \qquad (n=1, 2, \cdots)$$

converges in probability to $P(E)$, as $n\to+\infty$.

Examples, showing that the class of the mixing sequences of σ-algebras is not empty, are given e.g. in papers [1], [2], where concrete mixing sequences of events in the sense of the definition of A. Rényi have been studied.

Rosenblatt ([3]) introduced the notion of the strongly mixing and Kolmogorov ([6]) used the notion of the completely regular sequences of σ-algebras. Both notions require more than that of the regular sequences of σ-algebras. A sequence $\mathscr{G}_n \supset \mathscr{G}_{n+1}$ $(n=1, 2, \cdots)$ of σ-algebras is called regular if the σ-algebra $\bigcap_{n=1}^{\infty} \mathscr{G}_n$ is a trivial one. (The trivial σ-algebra is that σ-algebra which contains only sets with measure 0 or 1.) In this note we shall show that the properties of mixing sequences of σ-algebras are close to those of regular sequences.

2. First we shall show the following

Theorem 1. *Let $E\in\mathscr{A}$ be an arbitrary fixed event and let $\{\mathscr{G}_n\}$ be a mixing sequence of σ-algebras. Then*

(4) $$\lim_{n\to\infty} \sup_{B\in\mathscr{G}_n} |P(EB)-P(E)P(B)| = 0.$$

Proof. It can be easily seen that

$$\sup_{B\in\mathcal{G}_n} |P(EB)-P(E)P(B)| = \sup_{B\in\mathcal{G}_n} \left|\int_B (P(E|\mathcal{G}_n)-P(E))\,dP\right| \leqq$$

$$\leqq \sup_{B\in\mathcal{G}_n} \int_B |P(E|\mathcal{G}_n)-P(E)|\,dP \leqq \int_\Omega |P(E|\mathcal{G}_n)-P(E)|\,dP.$$

We have $0 \leqq P(E|\mathcal{G}_n) \leqq 1$ with probability 1. Since the sequence $\{\mathcal{G}_n\}$ is mixing we obtain (4) from (3) by LEBESGUE's convergence theorem.

Remark. If, for $n=1, 2, \cdots, \mathcal{G}_n \supset \mathcal{G}_{n+1}$, then relation (4) is the necessary and sufficient condition for $\{\mathcal{G}_n\}$ to be a regular sequence of σ-algebras. (See e.g. [5].)

It is known that the set-theoretical intersection of σ-algebras is also σ-algebra. By the union of σ-algebras we mean that σ-algebra which is generated by the set-theoretical union of these σ-algebras. It is obvious that the union of trivial σ-algebras is their set-theoretical union. We mean by the "limes inferior" of a sequence of σ-algebras the σ-algebra

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathcal{G}_k$$

and we denote it by $\liminf_{n\to\infty} \mathcal{G}_n$.

Theorem 2. *Let $\{\mathcal{G}_n\}$ be a mixing sequence of σ-algebras. Then* $\liminf_{n\to+\infty} \mathcal{G}_n$ *is a trivial σ-algebra.*

Proof. It is enough to show that for $n=1, 2, \cdots$ the σ-algebras $\bigcap_{k=n}^{\infty} \mathcal{G}_k$ are trivial. Applying assertion (4) of Theorem 1 to the event $E \in \bigcap_{k=n}^{\infty} \mathcal{G}_k$ (and writing E instead of B in (4)) we obtain $P(E)=(P(E))^2$. From this our assertion follows.

The following theorem asserts that regular sequences of σ-algebras are mixing.

Theorem 3. *Let $\{\mathcal{G}_n\}$ be a monotonically decreasing sequence of σ-algebras, i.e. for $n=1, 2, \cdots$ let $\mathcal{G}_n \supset \mathcal{G}_{n+1}$. In order that the sequence $\{\mathcal{G}_n\}$ be mixing, it is necessary and sufficient that the sequence $\{\mathcal{G}_n\}$ be regular.*

Proof. If the sequence $\{\mathcal{G}_n\}$ is mixing then, by Theorem 2, $\liminf_{n\to\infty} \mathcal{G}_n = \bigcap_{n=1}^{\infty} \mathcal{G}_n$ is a trivial σ-algebra.

Conversely, if for $n=1, 2, \cdots$ we have $\mathcal{G}_n \supset \mathcal{G}_{n+1}$, then for every fixed event E the sequence

$$P(E|\mathcal{G}_n) \quad (n=1, 2, \cdots)$$

is a martingale which converges with probability 1 (and, consequently, in probability measure, too) to the random variable

$$P(E|\bigcap_{n=1}^{\infty} \mathcal{G}_n).$$

(See e.g. Doob [4], Section VII, Theorem 4. 3.) Since in our case $\bigcap_{n=1}^{\infty} \mathscr{G}_n$ is a trivial σ-algebra, the last conditional probability is equal to $P(E)$ with probability 1. This proves our assertion.

3. Let $\{\mathscr{G}_n\}$ be a sequence of σ-algebras. Further consider all the random variables defined on Ω and measurable with respect to the σ-algebra $\mathscr{A}$ which are square integrable. If $\mathscr{H}$ denotes the set of these random variables, then for $\xi(\omega) \in \mathscr{H}$ $\eta(\omega) \in \mathscr{H}$ we define the inner product of ξ and η by

$$\int_{\Omega} \xi \cdot \eta \, dP,$$

and we denote it by (ξ, η). The norm of a random variable ξ is defined as $(\xi, \xi)^{\frac{1}{2}}$ and it is denoted by $\|\xi\|$. Then $\mathscr{H}$ is a Hilbert space with the inner product (ξ, η) Let $E \in \mathscr{G}_n$ $(n=1, 2, \cdots)$ and consider the linear combinations of the random variables $\chi_E - P(E)$, further consider also the limits in the mean of these linear combinations. The set of these random variables will be denoted by $\mathscr{H}_1$. Clearly, $\mathscr{H}_1$ is a subspace of the Hilbert space $\mathscr{H}$. Let us also consider that subspace $\mathscr{H}_2$ of $\mathscr{H}$ which is orthogonal to $\mathscr{H}_1$.

We prove now the following assertion which is well known. We prove it only for the sake of completeness.

Lemma *If $\xi \in \mathscr{H}_2$ is a random variable, then we have for every n*

$$P\big(M(\xi|\mathscr{G}_n) = M(\xi)\big) = 1.$$

Proof. It is enough to prove this assertion in case that $M(\xi)=0$. Let A be the event $\{\omega: M(\xi|\mathscr{G}_n) > \varepsilon\}$, where ε is an arbitrary fixed positive number. Then, since $M(\xi|\mathscr{G}_n)$ is $\mathscr{G}_n$-measurable, we have $A \in \mathscr{G}_n$. Further

$$\int_A M(\xi|\mathscr{G}_n)\, dP = \int_{\Omega} M(\xi\chi_A|\mathscr{G}_n)\, dP = \int_{\Omega} \xi\chi_A \, dP.$$

Since, by our supposition, ξ and $\chi_A - P(A)$ are orthogonal, we obtain that

$$\int_A M(\xi|\mathscr{G}_n)\, dP = P(A) M(\xi) = 0.$$

On the other hand

$$0 = \int_A M(\xi|\mathscr{G}_n)\, dP \geqq \varepsilon P(A) \geqq 0.$$

This results that $P(A)=0$. Let further B be the event $\{\omega: M(\xi|\mathscr{G}_n) < -\varepsilon\}$ $(\varepsilon > 0)$. Then $B \in \mathscr{G}_n$ and we obtain in such a way as above that

$$\int_B M(\xi|\mathscr{G}_n)\, dP = 0.$$

On the other hand

$$0 = \int_B M(\xi|\mathscr{G}_n)\,dP = -\varepsilon P(B) \leqq 0.$$

So we have $P(B)=0$. Since

$$P(|M(\xi|\mathscr{G}_n)| > \varepsilon) = P(A)+P(B) = 0,$$

and $\varepsilon > 0$ was chosen arbitrarily, we proved that

$$M(\xi|\mathscr{G}_n) = 0, \qquad (n = 1, 2, \cdots)$$

with probability 1.

We shall use this Lemma in the following assertion which facilitates to decide whether a sequence $\{\mathscr{G}_n\}$ of σ-algebras is mixing or not.

Theorem 4. *Let $\{\mathscr{G}_n\}$ be a sequence of σ-algebras. A necessary and sufficient condition for $\{\mathscr{G}_n\}$ to be mixing is that for every fixed $E \in \mathscr{G}_k$ $(k=1, 2, \cdots)$ the sequence*

$$P(E|\mathscr{G}_n) \qquad (n = 1, 2, \cdots)$$

of random variables converge in probability to $P(E)$.

Proof. The necessity part of the assertion is obvious. The sufficiency part of the proof can be performed as follows. Let $\varepsilon > 0$ be an arbitrary fixed number and let E be an arbitrary event. Then

$$\varepsilon P(|P(E|\mathscr{G}_n) - P(E)| > \varepsilon) \leqq M(|M((\chi_E - P(E))|\mathscr{G}_n)|). \tag{5}$$

Let us decompose $\chi_E - P(E)$ in the form

$$\xi_1 + \xi_2$$

where $\xi_1 \in \mathscr{H}_1$ and $\xi_2 \in \mathscr{H}_2$, $\mathscr{H}_1$ and $\mathscr{H}_2$ being defined as above. Since $M(\chi_E - P(E))=0$, further $M(\xi_1)=0$, one has $M(\xi_2)=0$. So by our Lemma we have for every n

$$M(\xi_2|\mathscr{G}_n) = 0$$

with probability 1. On the other hand

$$M((\chi_E - P(E))|\mathscr{G}_n) = M(\xi_1|\mathscr{G}_n) + M(\xi_2|\mathscr{G}_n) = M(\xi_1|\mathscr{G}_n)$$

with probability 1. So we have

$$M(|M((\chi_E - P(E))|\mathscr{G}_n)|) = M(|M(\xi_1|\mathscr{G}_n)|). \tag{6}$$

ξ_1, being element of $\mathscr{H}_1$, can be approximated in the mean by finite linear combinations of the elements $\chi_A - P(A)$, $(A \in \mathscr{G}_k,\ k=1, 2, \cdots)$. Denote the sequence, approximating ξ_1 in the mean, by $\eta_1, \eta_2, \cdots$ $(\eta_j \in \mathscr{H}_1)$. For every fixed k the sequence

$$M(\eta_k|\mathscr{G}_n) \qquad (n = 1, 2, \cdots)$$

converges, obviously, in probability to 0. Let $\delta > 0$ be arbitrary and let us put k such that $\|\xi_1 - \eta_k\| < \delta$ be satisfied. Then fix k. It is easily seen that η_k is bounded and so is $M(\eta_k|\mathscr{G}_n)$ with probability 1. Now we have

$$M(|M(\xi_1|\mathscr{G}_n)|) \leqq M(|M((\xi_1 - \eta_k)|\mathscr{G}_n)|) + M(|M(\eta_k|\mathscr{G}_n)|). \tag{7}$$

The second member on the right-hand side of (7) by LEBESGUE'S theorem converges to 0, while the first is smaller than

$$M(|M((\xi_1 - \eta_k)|\mathscr{G}_n)|) \leqq M(M(|\xi_1 - \eta_k||\mathscr{G}_n)) \leqq \|\xi_1 - \eta_k\|.$$

Conferring (7), (6) and (5) we obtain that

$$\limsup_{n \to \infty} P(|P(E|\mathscr{G}_n) - P(E)| > \varepsilon) \leqq \frac{\delta}{\varepsilon}. \tag{8}$$

Since $\varepsilon > 0$ and $\delta > 0$ vary independently each of other, (8) means the assertion of the theorem.

Theorem 4 gives similar conditions for $\{\mathscr{G}_n\}$ to be mixing as the conditions of the theorem of A. RÉNYI ([1]) for the sequence of events $\{B_n\}$ to be mixing with density $d\,(0 < d < 1)$.

Theorem 5. *Let $\{\mathscr{G}_n\}$ be a mixing sequence of σ-algebras and let z be a random variable having finite mean-value. Then the sequence*

$$M(z \mid \mathscr{G}_n) \qquad (n = 1, 2, \cdots)$$

of random variables converges in probability to $M(z)$.

Proof. The assertion of the theorem is true if z is of the form:

$$\sum_{k=1}^{j} c_k \chi_{E_k},$$

where c_k $(k = 1, 2, \cdots, j)$ is a real number, $E_k \in \mathscr{A}$ $(k = 1, 2, \cdots, j)$ are events such that $E_k \cap E_l = \varnothing$, and $\bigcup_{k=2}^{j} E_k = \Omega$, further χ_{E_k} denotes the indicator of E_k and j is finite positive integer. Since $M(z)$ is finite, the random variable z can be approximated in L^1 norm by the random variables of the mentioned form as close as we please. Let z^* be such a random variable for which

$$M(|z^* - z|) < \varepsilon$$

holds. Then

$$\int_{\Omega} |M(z|\mathcal{G}_n) - M(z)|\, dP \leqq \int_{\Omega} |M(z|\mathcal{G}_n) - M(z^*|\mathcal{G}_n)|\, dP +$$

$$+\int_{\Omega} |M(z^*|\mathcal{G}_n) - M(z^*)|\, dP + \int_{\Omega} |z^* - z|\, dP \leqq$$

$$\leqq \int_{\Omega} M(|z-z^*||\mathcal{G}_n)\, dP + \int_{\Omega} |M(z^*|\mathcal{G}_n) - M(z^*)| dP + M(|z-z^*|).$$

The first and the third terms on the right hand side of this inequality are smaller than ε and the second converges to zero. This proves the theorem.

4. It is interesting to investigate the analogon of Theorem 4 in case of the almost everywhere convergence. Theorem 6 makes this for martingales.

Theorem 6. *Let $\{\mathcal{G}_n\}$ be a sequence of σ-algebras and suppose that for every event E the conditional probabilities $P(E|\mathcal{G}_n)$ $(n=1, 2, \cdots)$ form a martingale. If for every fixed $E \in \mathcal{G}_k$ $(k=1, 2, \cdots)$ we have*

$$P\big(\lim_{n\to+\infty} P(E|\mathcal{G}_n) = P(E)\big) = 1,$$

then the same holds for every event E.

Proof. We have for arbitrary fixed E

$$0 \leqq P(E|\mathcal{G}_n) \leqq 1.$$

Thus by the convergence theorem of the martingales (cf. Doob [4], Section VII, Theorem 4. 1.) the limit

$$\lim_{n\to+\infty} P(E|\mathcal{G}_n) = \xi_E(\omega),$$

exists with probability 1, where $\xi_E(\omega)$ is a random variable. We have further $M(\xi_E(\omega)) = P(E)$. So it remains to prove that $P(\xi_E(\omega) = P(E)) = 1$. Let us consider for this purpose $M(|\xi_E(\omega) - P(E)|)$. We have

$$M(|\xi_E(\omega) - P(E)|) \leqq M(|\xi_E(\omega) - P(E|\mathcal{G}_n)|) + M(|M((\chi_E - P(E))|\mathcal{G}_n)|). \tag{9}$$

By Lebesgue's theorem, the first term on the right hand side converges to zero as $n \to \infty$. For dealing with the second member, let us decompose the random variable $\chi_E - P(E)$ into the form

$$\xi_1 + \xi_2,$$

where $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$; $\mathcal{H}_1$ and $\mathcal{H}_2$ being defined as above. Since $\xi_1 \in \mathcal{H}_1$, one has $M(\xi_1) = 0$ and so $M(\xi_2) = 0$. By our Lemma we have for every n

$$M(\xi_2|\mathcal{G}_n) = 0.$$

with probability 1. On the other hand

$$M((\chi_E - P(E))|\mathcal{G}_n) = M(\xi_1|\mathcal{G}_n) + M(\xi_2|\mathcal{G}_n) = M(\xi_1|\mathcal{G}_n)$$

with probability 1. So

(10) $$M(|M((\chi_E - P(E))|\mathcal{G}_n)|) = M(|M(\xi_1|\mathcal{G}_n)|).$$

ξ_1, being element of $\mathcal{H}_1$, can be approximated in the mean by finite linear combinations of the elements $\chi_A - P(A)$ $(A \in \mathcal{G}_k,\ k = 1, 2, \cdots)$. Denote this sequence, approximating ξ_1 in the mean, by $\eta_1, \eta_2, \cdots$. For every fixed k we have with probability 1

$$\lim_{n \to +\infty} M(\eta_k|\mathcal{G}_n) = 0.$$

Let $\varepsilon > 0$ be arbitrary and let us put k such that $\|\xi_1 - \eta_k\| < \varepsilon$ be satisfied. Then fix k. It is obvious that η_k is bounded and so is $M(\eta_k|\mathcal{G}_n)$. Now we have

(11) $$M(|M(\xi_1|\mathcal{G}_n)|) \leqq M(|M((\xi_1 - \eta_k|\mathcal{G}_n)|) + M(|M(\eta_k|\mathcal{G}_n)|).$$

The second member on the right hand side of (11) converges to 0, while the first is smaller than $\|\xi_1 - \eta_k\|$. Conferring (11), (10), and (9) we see that

(12) $$\begin{aligned} M(|\xi_E(\omega) - P(E)|) &\leqq \liminf_{n\to\infty} M(|\xi_E(\omega) - P(E|\mathcal{G}_n)|) + \\ &+ \liminf_{n\to\infty} M(|M((\xi_1 - \eta_k)|\mathcal{G}_n)|) + \liminf_{n\to\infty} M(|M(\eta_k|\mathcal{G}_n)|) \leqq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, the inequality (12) means our assertion.

5. By the aid of the mixing sequences of σ-algebras sequences of random events, which are mixing with density $d\,(0 < d < 1)$ can be constructed as follows:

Theorem 7. *Let $\{\mathcal{G}_n\}$ be a mixing sequence of σ-algebras and $\{B_n\}$ a sequence of random events, for which $B_n \in \mathcal{G}_n$, further $\lim_{n\to+\infty} P(B_n) = d$ exists. Then $\{B_n\}$ is a mixing sequence of events with density d.*

Proof. Let E be an arbitrary event. By our supposition the condition of Theorem 1 is satisfied. So we have

$$|P(EB_n) - dP(E)| \leqq \sup_{B \in \mathcal{G}_n} |P(EB) - P(E)P(B)| + P(E)|P(B_n) - d|.$$

Letting $n \to \infty$, the limit of the right hand side will be 0. This proves the theorem.

6. Consider now some consequences of the above results. We say that a sequence $\{\zeta_n\}$ $(n = 1, 2, \cdots)$ of random variables is mixing if the sequence of the corresponding σ-algebras $\mathcal{G}_n$ $(n = 1, 2, \cdots)$ generated by the random variable ζ_n is mixing.

Theorem 8. *If the sequence ζ_n $(n = 1, 2, \cdots)$ of random variables is mixing and η is an arbitrary random variable the ζ_n is asymptotically independent of η. If,* *in addition, ζ_n converges in probability to a random variable ζ, then ζ is constant with probability* 1.

Proof. The first assertion follows immediately from Theorem 1. In fact, x and y being arbitrary real numbers, by (4) we obtain if $n\to\infty$

$$|P(\zeta_n < x, \eta < y) - P(\zeta_n < x)P(\eta < y)| \leqq$$
$$\leqq \sup_{B\in\mathcal{G}_n} |P(B, \eta < y) - P(B)P(\eta < y)| \to 0.$$

From this it follows especially that if ζ_n converges in probability to ζ, then for every real x

$$\lim_{n\to+\infty} P(\zeta_n < x, \zeta < x) = (P(\zeta < x))^2.$$

On the other hand, if $\varepsilon > 0$ is an arbitrary number

$$P(\zeta_n < x, \zeta < x) = P(\zeta_n < x, \zeta < x, |\zeta_n - \zeta| < \varepsilon) + P(\zeta_n < x, \zeta < x, |\zeta_n - \zeta| \geqq \varepsilon).$$

The second member on the right hand side converges to 0, while the first satisfies the inequality

$$P(\zeta_n < x - \varepsilon, |\zeta_n - \zeta| < \varepsilon) \leqq P(\zeta_n < x, \zeta < x\, |\zeta_n - \zeta| < \varepsilon) \leqq P(\zeta_n < x).$$

If x and $x - \varepsilon$ are continuity points of the distribution function of ζ, then the right hand side converges to $P(\zeta < x)$ and the lim inf of the left hand side of the inequality is greater than $P(\zeta < x - \varepsilon)$. Since $\varepsilon > 0$ was chosen arbitrarily, we see that

$$\lim_{n\to\infty} P(\zeta_n < x, \zeta < x) = P(\zeta < x).$$

So we have

$$P(\zeta < x) = (P(\zeta < x))^2,$$

which means that for every real x

$$P(\zeta < x) = 0, \quad \text{or} \quad 1.$$

This proves our assertion.

Theorem 9. *Let $\{\mathcal{G}_n\}$ be a mixing sequence of σ-algebras in the probability space $\{\Omega, \mathcal{A}, P\}$. If Q is another probability measure, defined on $\mathcal{A}$, and it is absolutely continuous with respect to P, then for every event E the sequence*

$$Q(E|\mathcal{G}_n) \quad (n = 1, 2, \cdots)$$

of conditional Q-probabilities converges in P (and, consequently, in Q)-probability to $Q(E)$, as $n\to\infty$.

Proof. $Q(E|\mathcal{G}_n)$, being conditional Q-probability, is a random variable, which is measurable with respect to $\mathcal{G}_n$ and for every $A\in\mathcal{G}_n$ we have

$$Q(EA) = \int_A Q(E|\mathcal{G}_n)\, dQ = \int_A Q(E|\mathcal{G}_n)\lambda(\omega)\, dP,$$

where $\lambda(\omega)$ is the Radon—Nikodým derivative of Q with respect to P. Now we have

$$Q(EA) = \int_A Q(E|\mathcal{G}_n)\lambda\, dP = \int_A M\big(Q(E|\mathcal{G}_n)\lambda|\mathcal{G}_n\big)\, dP = \int_A Q(E|\mathcal{G}_n) M(\lambda|\mathcal{G}_n)\, dP.$$

On the other hand for every $A \in \mathcal{G}_n$

$$Q(EA) = \int_A \chi_E \lambda\, dP = \int_A M(\chi_E \lambda|\mathcal{G}_n)\, dP.$$

Since the conditional expectation is uniquely determined mod P, we have with probability 1

$$Q(E|\mathcal{G}_n) M(\lambda|\mathcal{G}_n) = M(\chi_E\lambda|\mathcal{G}_n).$$

By Theorem 5 the random variables

$$M(\lambda|\mathcal{G}_n) \quad \text{and} \quad M(\chi_E \lambda|\mathcal{G}_n)$$

converge in probability to 1 and to

$$M(\chi_E \lambda) = \int_E \lambda\, dP = Q(E),$$

respectively. From this and from the preceding equality our theorem follows.

Corollary. *If $A_n \in \mathcal{G}_n$, $\lim_{n\to+\infty} P(A_n) = d$, then under the conditions of Theorem 9 we have for every event E*

$$\lim_{n\to\infty} Q(A_n E) = dQ(E),$$

i.e., if a sequence $\{A_n\}$ is mixing with density d in the probability space $\{\Omega, \mathcal{A}, P\}$, then it is mixing with the same density in $\{\Omega, \mathcal{A}, Q\}$ provided that Q is absolutely continuous probability measure with respect to P.

Proof. By Theorem 9, $Q(E|\mathcal{G}_n)$ converges in probability to $Q(E)$ and so

$$Q(A_n E) = \int_{A_n} Q(E|\mathcal{G}_n)\, dQ \to Q(E) \lim_{n\to\infty} P(A_n) = dQ(E),$$

as $n\to\infty$, because $\lim_{n\to\infty} Q(A_n) = \lim_{n\to\infty} \int_{A_n} M(\lambda|\mathcal{G}_n) dP = \lim_{n\to\infty} P(A_n) = d$.

As another consequence of Theorem 9 we prove now

Theorem 10. *Let $\{\mathcal{G}_n\}$ be a mixing sequence of σ-algebras in the probability space $\{\Omega, \mathcal{A}, P\}$ and $\{\zeta_n\}$ a sequence of random variables such that ζ_n is $\mathcal{G}_n$-measurable $(n=1, 2, \cdots)$. Let further Q be a probability measure which is absolutely continuous with respect to P. If*

$$\lim_{n\to+\infty} P(\zeta_n < x) = F(x),$$

where $F(x)$ is a distribution function and the limit relation holds for every fixed x which is a continuity point of $F(x)$, then we have at every continuity point of $F(x)$

$$\lim_{n\to+\infty} Q(\zeta_n < x) = F(x).$$

Remark. Theorem 10 is a generalization of Theorem 3.1 of [5] and of the corresponding theorem of [1], where a similar assertion has been proved for regular sequences of σ-algebras.

Proof. Let x be an arbitrary fixed continuity point of $F(x)$. Then the event $A_n = \{\omega : \zeta_n(\omega) < x\}$ belongs to $\mathcal{G}_n$. So by the Corollary to Theorem 9 (putting Ω instead of E) we obtain our assertion.

References

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