On oscillation of the number of primes in an arithmetical progression

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1. J. E. LITTLEWOOD [1] proved — in the contrary to an assertion of RIEMANN — that, for a suitable sequence $x_1' < x_2' < \dots$ of integers, the inequality

$$\pi(x_\nu') > \operatorname{li} x_\nu'$$

holds. SKEWES [2] has obtained an upper bound for the first x for which the difference $\pi(x) - \operatorname{li} x$ is positive, namely $\exp\exp\exp\exp(7{,}705)$. Later S. KNAPOWSKI [3] — using the ideas of P. TURÁN — gave another proof of this fact. In the last year, S. LEHMAN [4] gave a better upper bound, namely $1{,}65 \cdot 10^{1165}$.

Recently S. KNAPOWSKI and P. TURÁN gave an explicit, localized $\Omega_\pm$ estimation for the difference $\pi(x, 4, 1) - \frac{1}{2} \operatorname{li} x$, where, in general, $\pi(x, k, l)$ denotes the number of primes in the arithmetical progression $\equiv l \pmod k$ not exceeding x.

The investigation of the oscillation behavior of $\pi(x, 4,3) - \frac{1}{2} \operatorname{li} x$ is a simpler case. However, for this we need another method.

In the following, $c, c_0, c_1, \dots, \delta$ will denote explicitly calculable numerical constants (e. c. n. c.), not the same in every place. $e_1(x)$ means e^x and $e_\nu(x) = e_1(e_{\nu-1}(x))$, further $\log_1 x = \log x$ and $\log_\nu x = \log(\log_{\nu-1} x)$. Throughout the paper the letter p is preserved for primes.

We shall prove the following

Theorem 1. *For every $T > c_0$ we have*

$$\max_{T \leqq x \leqq T^\varkappa} \frac{\pi(x, 4, 3) - \frac{1}{2} \operatorname{li} x}{\sqrt{x}/\log x} > \delta, \qquad \min_{T \leqq x \leqq T^\varkappa} \frac{\pi(x, 4, 3) - \frac{1}{2} \operatorname{li} x}{\sqrt{x}/\log x} < -\delta,$$

where δ and c_0 are e. c. n. positive constants and $\varkappa = (2 + \sqrt{3})^2$.

In their papers [5], [6] KNAPOWSKI and TURÁN dealt with the oscillation behavior of the functions

$$a(x) = \sum_{p \equiv l_1 \pmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l_2 \pmod 8} \log p \cdot e^{-px},$$

$$b(x) = \sum_{p \equiv l_1 \pmod 8} e^{-px} - \sum_{p \equiv l_2 \pmod 8} e^{-px}.$$

In [5] they proved that if $0<\delta<c_1$, then for $l_1 \not\equiv l_2 \not\equiv 1 \pmod 8$ we have

$$\max_{\delta \leqq x \leqq \delta^{1/3}} a(x) > \frac{1}{\sqrt{\delta}} e_1 \left(-22 \frac{\log(1/\delta) \cdot \log_3(1/\delta)}{\log_2(1/\delta)}\right),$$

where c_1 is an e. c. n. c. In [6] they proved that for $l_1, l_2 = 3, 5, 7$ $(l_1 \neq l_2)$ we have

$$\max_{\delta \leqq x \leqq \delta^{1/3}} |b(x)| \geqq \frac{1}{\sqrt{\delta}} e_1 \left(-23 \frac{\log(1/\delta) \cdot \log_3(1/\delta)}{\log_2(1/\delta)}\right).$$

The authors remarked: "To the more difficult problem of one-sided theorems (for $b(x)$) we hope to return." This problem seems still to be open.

From our Theorem 5 it follows that, for $0<y<c$ and for all $l_1 \not\equiv l_2 \not\equiv 1 \pmod 8$, the inequality

$$\max_{y^{\varkappa} \leqq x \leqq y} b(x) \sqrt{x} \log(1/x) > \delta$$

holds, where $\varkappa = (2+\sqrt{3})^2$.

We formulate now some theorems the proofs of which are similar to the proof of Theorem 1.

Let $N_k(l)$ denote the number of solutions of the congruence $x^2 \equiv l \pmod k$. For the moduli k in

(A) 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 24,

the position of the zeros of $L(s, \chi)$ for all $\chi \pmod k$ is known in the neighbourhood of the real line. Especially, it was proved by Haselgrove, that the $L(s, \chi)$ are non-vanishing on the real line in the critical strip.

Theorem 2. *For k in* (A) *and for all of those pairs l_1, l_2 for which $N_k(l_1) = N_k(l_2)$, $l_1 \not\equiv l_2 \pmod k$, we have*

$$\max_{T \leqq x \leqq T^{\varkappa}} \frac{\pi(x,k,l_1) - \pi(x,k,l_2)}{\sqrt{x}/\log x} > \delta, \qquad \min_{T \leqq x \leqq T^{\varkappa}} \frac{\pi(x,k,l_1) - \pi(x,k,l_2)}{\sqrt{x}/\log x} < -\delta,$$

if $T>c$, where $\varkappa = (2+\sqrt{3})^2$, c and δ are e. c. n. positive constants.

Theorem 3. *For all k in* (A) *and for all l for which $N_k(l) = 0$, we have the inequalities*

$$\max_{T \leqq x \leqq T^{\varkappa}} \frac{\pi(x,k,l) - \dfrac{\operatorname{li} x}{\varphi(k)}}{\sqrt{x}/\log x} > \delta, \qquad \min_{T \leqq x \leqq T^{\varkappa}} \frac{\pi(x,k,l) - \dfrac{\operatorname{li} x}{\varphi(k)}}{\sqrt{x}/\log x} < -\delta,$$

whenever $T>c$, where δ and c are e. c. positive numerical constants.

Let

$$\sigma(x,k,l) = \sum_{p\equiv l \,(\mathrm{mod}\, k)} e^{-p/x}, \quad s(x) = \sum_{n=2}^{\infty} \frac{1}{\log n} e^{-n/x}.$$

The following assertions hold.

Theorem 4. *For every k in* (A) *and for all l for which* $N_k(l)=0$, *we have*

$$\max_{T\leqq x\leqq T^{\varkappa}} \frac{\sigma(x,k,l)-\dfrac{s(x)}{\varphi(k)}}{\sqrt{x}/\log x} > \delta, \quad \min_{T\leqq x\leqq T^{\varkappa}} \frac{\sigma(x,k,l)-\dfrac{s(x)}{\varphi(k)}}{\sqrt{x}/\log x} < -\delta$$

if $T>c$, *where* $\varkappa=(2+\sqrt{3})^2$, *further c and δ are e. c. positive numerical constants.*

Theorem 5. *For every k in* (A) *and all* l_1, l_2 *for which* $N_k(l_1)=N_k(l_2)$, $l_1\not\equiv l_2$ (mod k), *we have*

$$\max_{T\leqq x\leqq T^{\varkappa}} \frac{\sigma(x,k,l_1)-\sigma(x,k,l_2)}{\sqrt{x}/\log x} > \delta,$$

if $T>c$, *where* $\varkappa=(2+\sqrt{3})^2$, *further c and δ are positive e. c. n. c.*

The method of the proofs of our Theorems is the same as was elaborated for the omega-estimation of $M(x)=\sum_{n\leqq x}\mu(n)$ in my dissertation [7] and in the paper [8]. However, we use here an idea of Rodosskу in a deeper form [9].

2. Some lemmas.

Lemma 1. *If*

$$F(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}$$

is absolutely convergent for $\sigma\geqq\sigma_0$, *then*

$$\sum_{n=1}^{\infty} a_n e_1\left(-\frac{\log^2 n}{4u}\right) = \frac{i\sqrt{u}}{\sqrt{\pi}} \int_{(\sigma_0)} F(w)\, e_1(w^2 u)\, dw. \tag{2.1}$$

For the proof see [9].

Lemma 2. [9] *For* $0<\alpha\leqq 1$ *and* $u\to\infty$,

$$\frac{1}{2u}\int_1^{\infty} x^{\alpha-1}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right) dx = 2\sqrt{\pi u}\, e_1(\alpha^2 u)+O(1). \tag{2.2}$$

Lemma 3. [9] *Let* $u \geqq 1$ *and* y, z *be defined by*

(2.3)-(2.4) $$\log y = 2u\left(1-\frac{\sqrt{3}}{2}\right), \quad \log z = 2u\left(1+\frac{\sqrt{3}}{2}\right).$$

The following inequalities hold:

(2.5) $$\frac{1}{2u}\int_1^y x^{-1/2}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx < ce_1\left(\frac{u}{4}\right),$$

(2.6) $$\frac{1}{2u}\int_z^\infty x^{-1/2}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx < ce_1\left(\frac{u}{4}\right).$$

Lemma 4. *Let*

(2.7) $$R(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log\left(w-\frac{1}{2}\right)e_1(w^2u)\,dw.$$

Then

$$|R(u)| > \frac{c}{\sqrt{u}}e_1(u/4), \qquad \text{if } u > c_1.$$

Proof. Using the well-known formula

$$\log s = \int_0^\infty \frac{e^{-v}-e^{-sv}}{v}\,dv \quad (\operatorname{Re} s > 0)$$

due to EULER, we obtain that

$$R(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\int_0^\infty \frac{e_1(-v)-e_1(-(w-\frac{1}{2})v)}{v}\,dv\,e_1(w^2u)\,dw =$$

$$= \pi\int_0^\infty\left[-e_1(-v)+e_1\left(-\frac{v^2}{4u}+\frac{v}{2}\right)\right]\frac{dv}{v}.$$

Since

$$\left|\int_0^1\left[e_1\left(-\frac{v}{4u}+\frac{v}{2}\right)-e_1(-v)\right]\frac{dv}{v}\right| < c \quad \text{and} \quad \int_1^\infty e_1(-v)\frac{dv}{v} < c,$$

the inequality

$$R_1(u) \overset{\text{def}}{=} \int_1^\infty e_1\left(-\frac{v^2}{4u}+\frac{v}{2}\right)\frac{dv}{v} = R(u)+O(1)$$

holds. Substituting $e_1(v)=x$ we obtain

$$R_1(u) = \int_e^\infty e_1\left(-\frac{\log^2 x}{4u}\right)\frac{x^{-1/2}}{\log x}\,dx,$$

thus

$$R_1(u) \geqq 2u(\log z)^{-2}\frac{1}{2u}\int_y^z e_1\left(-\frac{\log^2 x}{4u}\right)\log x\cdot x^{-1/2}\,dx \geqq$$

$$\geqq cu(\log z)^{-2}\sqrt{u}\,e_1(u/4) \geqq ce_1(u/4)\cdot u^{-1/2}, \quad c>0.$$

(See Lemmas 2, 3 and (2. 4).) Hence the assertion follows.

From Lemma 4 one can deduce the following

Lemma 5. *Let*

(2.8) $$J(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log\left(w-\tfrac{1}{2}\right)\Gamma(w)e_1(w^2u)\,dw.$$

Then

$$|J(u)| > ce_2(u/4)u^{-1/2}, \quad c>0.$$

Proof. Let L denote the broken line with vertices $1-i\infty$, $1-i\cdot 2$, $1/4-i\cdot 2$, $1/4+i\cdot 2$, $1+i\cdot 2$, $1+i\cdot\infty$. Let $\Gamma(\omega)=\Gamma(\tfrac{1}{2})+\varphi(\omega)$. So the inequalities

(2. 9) $$|\varphi(\omega)| \leqq c|w-1/2|, \qquad |\log(w-\tfrac{1}{2})\varphi(w)| < c|w-\tfrac{1}{2}|^{3/4}$$

hold on the line L. Let now

$$J(u) = \frac{i\sqrt{u}\,\Gamma(\tfrac{1}{2})}{\sqrt{\pi}}\int_{(2)}\log\left(w-\tfrac{1}{2}\right)e_1(w^2u)\,dw + \frac{i\sqrt{u}}{\sqrt{\pi}}\int_L \varphi(w)e_1(w^2u)\log\left(w-\tfrac{1}{2}\right)dw.$$

From (2. 9) it follows that the absolute value of second integral is majorized by $ce_1(u|4)u^{-1}$. For the first integral we use Lemma 4, and we obtain the assertion stated in Lemma 5.

3. Let us now introduce the following notations:

(3.1)-(3.2) $$f(s) = \sum_{p\equiv 3\,(\mathrm{mod}\,4)} p^{-s}; \quad g(s) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(\log n)^{-1}}{n^s};$$

(3.3) $$F(s) = f(s)-g(s) = \sum_{n=2}^{\infty}\frac{a_n}{n^s},$$

where the coefficients a_n of $F(s)$ are defined by

(3.4) $$a_n = \begin{cases} 1-\tfrac{1}{2}(\log n)^{-1}, & \text{if } n=p\equiv -1\,(\mathrm{mod}\,4), \ p \text{ prime}, \\ -\tfrac{1}{2}(\log n)^{-1} & \text{otherwise.}\end{cases}$$

Let $\zeta(s)$ be the Riemann zeta-function and let

$$L(s,\chi)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}.$$

We have evidently that

$$f(s)=\frac{1}{2}\log\frac{\zeta(s)}{L(s,\chi)}+h(s), \tag{3.5}$$

where $h(s)$ is a function represented by an absolutely convergent Dirichlet series in the halfplane $\operatorname{Re} s>1/3$ and hence regular.

Further we have $\frac{dg(s)}{ds}=-\zeta(s)$ and so $\frac{d}{ds}\big(g(s)+\log(s-1)\big)=-\zeta(s)+\frac{1}{s-1}$. Since the right hand side is an integral function, so is $g(s)+\log(s-1)$ regular on the whole plane. Hence it follows that $F(s)$ is regular at the point $s=1$. Further it is known that in the domain $0<\sigma<1, 0\leqq t\leqq 10,24$ the function $L(s,\chi)$ has a unique simple zero, namely at the point

$$\varrho=\tfrac{1}{2}+i\cdot 6{,}02\ldots=\tfrac{1}{2}+i\cdot\gamma. \tag{3.6}$$

In this domain $\zeta(s)$ is non-vanishing.

Let now

$$I(\tau)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\Gamma(w+i\tau)\,e_1(w^2u)\,dw, \tag{3.7}$$

where τ is a real number.

We shall now give an upper estimation for (3. 7) in the special cases $\tau=0$ and $\tau=\gamma$.

Let Γ denote the broken line with vertices $1,5-i\cdot\infty$; $1,5-4i$; $0,4-4i$; $0,4+4i$; $1,5+4i$; $1,5+i\cdot\infty$. For the estimation of $I(0)$ we transform the integration line in (3. 7) to Γ and we obtain

$$|I(0)|<ce_1(0{,}16\cdot u). \tag{3.8}$$

Choose now $\tau=\gamma$. Then the function $F(w+i\gamma)$ has a logarithmic singularity at the point $w=1/2$ and

$$F(w+i\gamma)=-\log(w-\tfrac{1}{2})+F_1(w),$$

where $F_1(w)$ is a regular function on the broken line Γ and on the right hand side of Γ. So we have

$$I(\gamma)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{\Gamma}F_1(w)\,e_1(w^2u)\,dw-\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log\left(w-\frac{1}{2}\right)e_1(w^2u)\,dw=P(u)-R(u).$$

For $P(u)$ we have the estimation $|P(u)| < ce_1(0, 16u)$. From Lemma 4

$$|R(u)| > \frac{ce_1(u/4)}{\sqrt{u}}$$

follows. So we have

(3.9) $$|I(\gamma)| > \frac{ce_1(u/4)}{\sqrt{u}}.$$

4. Let now

(4.1) $$A(x) = \sum_{n \leq x} a_n,$$

where the a_n are defined by (3. 4). It is evident, that

(4. 2) $$\pi(x, 4, 3) - \tfrac{1}{2}\,\mathrm{li}\,x = A(x) + O(1).$$

From Lemma 1 it follows that the $I(\tau)$ in (3. 7) can be represented as

$$I(\tau) = \sum_{n=2}^{\infty} a_n e_1\left(-\frac{\log^2 n}{4u} - i\tau \log n\right).$$

By partial integration follows:

(4.3) $$I(\tau) = \int_1^{\infty} A(x)\, x^{-1}\left(\frac{\log x}{2u} + i\tau\right) e_1\left(-\frac{\log^2 x}{4u} - i\tau \log x\right) dx.$$

Let further $I(\tau, 1, y)$, $I(\tau, y, z)$, $I(\tau, z, \infty)$ denote the integral on the right hand side extended for the intervals $[1, y]$, $[y, z]$, $[z, \infty]$, respectively. Let the values y, z be choosen as in (2. 3), (2. 4). Using the trivial estimation $|A(x)| < cx(\log x)^{-1}$ we have

$$|I(\tau, 1, y)| < c \int_2^{y} (\log x)^{-1}\left[\frac{\log x}{2u} + |\tau|\right] e_1\left(-\frac{\log^2 x}{4u}\right) dx \leqq$$

$$\leqq c(1+|\tau|) \int_2^{y} \frac{1}{\log x} e_1\left(-\frac{\log^2 x}{4u}\right) dx = c(1+|\tau|) \int_{\log 2}^{\log y} t^{-1} e_1\left(t - \frac{t^2}{4u}\right) dt$$

and by partial integration,

$$\int_2^{y} \frac{1}{\log x} e_1\left(-\frac{\log^2 x}{4u}\right) dx < cu^{-1} e_1(u/4).$$

Hence

(4.3) $$|I(\tau, 1, y)| < c(1+|\tau|)\, u^{-1} e_1(u/4)$$

follows. Using similar computations we obtain

$$|I(\tau, z, \infty)| < c(1+|\tau|)\, u^{-1}\, e_1(u/4). \tag{4.4}$$

Let now assume that for a fixed positive δ one of the inequalities

$$\max_{y \leqq x \leqq z} \left(A(x) - \delta \frac{x^{1/2}}{\log x} \right) \leqq 0, \tag{4.5}$$

$$\min_{y \leqq x \leqq z} \left(A(x) + \delta \frac{x^{1/2}}{\log x} \right) \geqq 0 \tag{4.6}$$

holds. Using this assumption we obtain such an inequality for $I(y)$ and $I(0)$ which contradicts (3. 8), (3. 9).

Indeed, we have

$$|I(\tau, y, z)| \leqq \left| \int_y^z \frac{A(x) \pm \delta \frac{x^{1/2}}{\log x}}{x} \left| \frac{\log x}{2u} + i\tau \right| e_1\left(-\frac{\log^2 x}{2u} \right) dx \right| +$$

$$+ \delta \int_y^z \frac{x^{-1/2}}{\log x} \left| \frac{\log x}{2u} + i\tau \right| e_1\left(-\frac{\log^2 x}{4u} \right) dx.$$

Using the inequality

$$\left| \frac{\log x}{2u} + i\tau \right| < c(1+|\tau|) \frac{\log x}{2u}$$

and our assumption, i.e. that one of the functions

$$A(x) \pm \delta \frac{x^{1/2}}{\log x}$$

has constant sign on the interval $[y, z]$, we have

$$|I(\tau, y, z)| \leqq c(1+|\tau|) I(0, y, z) + c\delta (1+|\tau|) \int_y^z \frac{x^{-1/2}}{\log x} e_1\left(-\frac{\log^2 x}{4u} \right) dx.$$

For the integral on the right hand side we have

$$\int_y^z \frac{x^{-1/2}}{\log x} e_1\left(-\frac{\log^2 x}{4u} \right) dx < \frac{c}{u} \int_1^\infty x^{-1/2} e_1\left(-\frac{\log^2 x}{4u} \right) dx < \frac{c}{\sqrt{u}} e_1(u/4)$$

by Lemma 2. Hence

$$|I(\tau, y, z)| < c(1+|\tau|)\, |I(0, y, z)| + c\delta(1+|\tau|)\, e_1(u/4) u^{-1}$$

and by (4. 3), (4. 4)

$$|I(\tau)| < c(1+|\tau|)\left\{|I(0)| + \delta\frac{e_1(u/4)}{\sqrt{u}} + \frac{e_1(u/4)}{u}\right\}. \tag{4.7}$$

Let now $\tau=\gamma$. Taking into account the inequalities (3. 8), (3. 9) we get

$$c_1 u^{-1/2} e_1(u/4) < c_2 e_1(0{,}16u) + \delta c_2 u^{-1/2} e_1(u/4) + c_3 u^{-1} e_1(u/4),$$

where $c_1>0$. This is impossible if $\delta<c_1/c_2$ and u is sufficiently large. Hence it follows that the inequalities cannot hold, i.e. we have

$$\max_{y\leq x\leq z}\frac{A(x)\log x}{\sqrt{x}} > \delta, \quad \min_{y\leq x\leq z}\frac{A(x)\log x}{\sqrt{x}} < -\delta,$$

if $u>c$.

Taking into account that

$$A(x) = \pi(x,4,3) - \tfrac{1}{2}\operatorname{li} x + O(1),$$

and that $z=y^{\varkappa}$ Theorem 1 follows.

5. In this section we give a sketch of Theorem 5 in the special case $k=8$ We shall use the following generalization of Lemma 1.

Lemma 6. *Let*

$$h(s) = \int_1^\infty x^{-s}\,dA(x) \tag{5.1}$$

absolutely and uniformly convergent in the halfplane $\sigma>\sigma_1(>0)$. Then

$$\int_1^\infty e_1\left(-\frac{\log^2 x}{4u}\right)dA(x) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(\sigma)} h(w)e_1(w^2u)\,dw. \tag{5.2}$$

The proof of this Lemma is very similar to that of Lemma 1 and so can be omitted.

Let l_1, l_2 be two different among the numbers 3, 5, 7, further let ε_p be defined by the relation

$$\varepsilon_p = \begin{cases} 1, & \text{if } p\equiv l_1 \pmod 8, \\ -1, & \text{if } p\equiv l_2 \pmod 8, \\ 0 & \text{otherwise,}\end{cases} \tag{5.3}$$

and let

$$g(s) = \sum_p \frac{\varepsilon_p}{p^s}, \quad s(x) = \sigma(x,8,l_1) - \sigma(x,8,l_2) = \sum_p \varepsilon_p e^{-p/x}. \tag{5.4)–(5.5}$$

Using a well-known relation we have

$$(5.6)\qquad \Gamma(s)g(s)=\int_0^\infty y^{s-1}\sum_p \varepsilon_p e^{-py}\,dy=\int_0^\infty \frac{s(x)}{x^{s+1}}\,dx=\int_0^1+\int_1^\infty = l(s)+h(s).$$

Here the function $l(s)$ is regular in the halfplane $\operatorname{Re} s=\sigma>0$ and $|l(s)|<c$ if $\sigma\geqq 1/10$, because $|s(x)|<c$ in the interval $0\leqq x\leqq 1$. Using now Lemma 6 with

$$(5.7)\qquad dA(x)=\frac{s(x)}{x^{1+i\tau}}\,dx,\quad h(s)=\int_1^\infty \frac{s(x)}{x^{s+1}}\,dx,$$

we obtain

$$(5.8)\qquad \int_1^\infty e_1\left(-\frac{\log^2 x}{4u}-i\tau\log x\right)s(x)\frac{dx}{x}=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)} h(w+i\tau)\,e_1(w^2u)\,dw.$$

Let us now introduce the following notations:

$$(5.9)\qquad I(\tau,a,b)=\int_a^b e_1\left(-\frac{\log^2 x}{4u}-i\tau\log x\right)s(x)\frac{dx}{x},$$

$$(5.10)\qquad K(\tau)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)} h(w+i\tau)\,e_1(w^2u)\,dw.$$

In the proof an essential role is played by some numerical data due to P. C. Haselgrove (see S. Knapowski and P. Turán [5], p. 254). Let

$$L(s,\chi_1)\overset{\text{def}}{=}\sum_{n=0}^\infty\left\{\frac{1}{(8n+1)^s}+\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}-\frac{1}{(8n+7)^s}\right\},$$

$$L(s,\chi_2)\overset{\text{def}}{=}\sum_{n=0}^\infty\left\{\frac{1}{(4n+1)^s}-\frac{1}{(4n+3)^s}\right\},$$

$$L(s,\chi_3)\overset{\text{def}}{=}\sum_{n=0}^\infty\left\{\frac{1}{(8n+1)^s}-\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}+\frac{1}{(8n+7)^s}\right\}.$$

Then in the domain

$$0<\sigma<1,\ |t|\leqq 12$$

the zeros of $L(s,\chi_1)$ are

$$\tfrac12\pm i\cdot 4{,}899\ldots,\quad \tfrac12\pm i\cdot 7{,}628\ldots,\quad \tfrac12\pm i\cdot 10{,}806\ldots$$

those of $L(s,\chi_2)$

$$\tfrac12\pm 2\cdot 6{,}020\ldots,\quad \tfrac12\pm 2\cdot 10{,}243\ldots,$$

and those of $L(s, \chi_3)$

$$\tfrac{1}{2} \pm i\cdot 3{,}576\ldots, \quad \tfrac{1}{2} \pm i\cdot 7{,}434\ldots, \quad \tfrac{1}{2} \pm i\cdot 9{,}503\ldots .$$

In particular, they are simple and different from each other.

We shall use that for the function $g(s)$ in (5. 4)

$$g(s) = \frac{-1}{4} \sum_{\chi \pmod 8} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \log L(s, \chi) + u(s), \tag{5.11}$$

where the function $u(s)$ has an absolutely convergent Dirichlet series representation in the halfplane $\sigma > \frac{1}{3}$, because 3, 5, 7 are quadratic non-residues mod 8. So we have

$$h(s) = -\frac{\Gamma(s)}{4} \sum_{\chi \pmod 8} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \log L(s, \chi) + v(s), \tag{5.12}$$

where $v(s)$ is a regular and bounded function in the strip $\frac{1}{3} < \sigma < 10$. Transforming the integration line in (5.10) to the broken line Γ (see (3.8)) we have

$$|K(0)| < ce_1(0{,}16u). \tag{5.13}$$

Choose $\tau = \gamma$ where $\frac{1}{2} + i\gamma$ is the first singularity of $g(s)$ in the upper halfplane ($\operatorname{Im} s > 0$). Using Lemma 5 instead of Lemma 4 we have

$$|K(\gamma)| > cu^{-1/2} e_1(u/4), \qquad c > 0. \tag{5.14}$$

Let now y, z be chosen as in (2. 3), (2. 4) and assume that one of the inequalities

$$\max_{y \le x \le z} \left(s(x) - \delta \frac{\sqrt{x}}{\log x} \right) \le 0, \quad \min_{y \le x \le z} \left(s(x) + \delta \frac{\sqrt{x}}{\log x} \right) \ge 0 \tag{5.15)-(5.16}$$

be satisfied with a positive δ. Using a similar argument as in the section 4, we can deduce from this assumption the inequality

$$|I(\tau, 1, \infty)| < c(1 + |\tau|)\{|I(0, 1, \infty)| + \delta u^{-1/2} e_1(u/4) + u^{-1} e_1(u/4)\}. \tag{5.17}$$

Taking into account that $I(\tau, 1, \infty) = K(\tau)$ and choosing $\tau = \gamma$, the inequality (5.17) contradicts the inequalities (5.13), (5.14) for a sufficiently small positive δ and for $u > c$. So the inequalities (5.15)—(5.16) for this δ cannot hold and hence the assertion follows.

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