# Quasitriangular operators

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Every square matrix with complex entries is unitarily equivalent to a triangular one. In other words, if $A$ is an operator on a finite-dimensional Hilbert space $H$, then there exists an increasing sequence $\{M_n\}$ of subspaces such that $\dim M_n = n$ $(n=0, \ldots, \dim H)$, and such that each $M_n$ is invariant under $A$. On a Hilbert space of dimension $\aleph_0$ the appropriate definition is this: $A$ is *triangular* if there exists an increasing sequence $\{M_n\}$ of finite-dimensional subspaces whose union spans $H$ such that each $M_n$ is invariant under $A$. It is easy, but not obviously desirable, to fill in the dimension gaps, and hence to justify the added assumption that $\dim M_n = n$ $(n=0, 1, 2, \ldots)$.

In many considerations of invariant subspaces $(AM \subset M)$ it is convenient to treat their projections instead $(AE=EAE)$. In terms of projections a necessary and sufficient condition that an operator $A$ on a separable Hilbert space $H$ be triangular is that

($\Delta$) *there exists an increasing sequence* $\{E_n\}$ *of projections of finite rank such that* $E_n \to 1$ *(strong topology) and such that* $AE_n - E_nAE_n = 0$ *for all* $n$.

This formulation suggests an asymptotic generalization of itself. An operator $A$ is *quasitriangular* if

($\Delta_1$) *there exists an increasing sequence* $\{E_n\}$ *of projections of finite rank such that* $E_n \to 1$ *(strong topology) and such that* $\|AE_n - E_nAE_n\| \to 0$.

(Informally: $E_n$ is approximately invariant under $A$.) The concept (but not the name) has been seen before; it plays a central role in the proofs of the Aronszajn—Smith theorem [1] on the existence of invariant subspaces for compact operators, and in the proofs of its various known generalizations [2], [3], [5].

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It is interesting and useful to examine a variant of the condition ($\Delta_1$); the variant requires that

($\Delta_2$) *there exists a sequence* $\{E_n\}$ *of projections of finite rank such that* $E_n \to 1$ *(strong topology) and such that* $\|AE_n - E_nAE_n\| \to 0$.

The only difference between $\Delta_1$ and $\Delta_2$ is that the latter does not require the sequence $\{E_n\}$ to be increasing.

There is still another pertinent condition. The set of all projections of finite rank, ordered by range inclusion, is a directed set. Since $E \to \|AE - EAE\|$ is a net on that directed set, it makes sense to say that

$$(\Delta_0) \qquad \liminf_{E \to 1} \|AE - EAE\| = 0.$$

What it means is that for every positive number $\varepsilon$ and for every projection $E_0$ of finite rank there exists a projection $E$ of finite rank such that $E_0 \leqq E$ and $\|AE - EAE\| < \varepsilon$.

The purpose of this paper is to initiate a study of quasitriangular operators. The study begins with the observation that approximately invariant projections that are large (in the sense of having large ranks) always exist (Section 1). The main result is the characterization of quasitriangular operators; it asserts (for separable spaces) that the conditions $\Delta_0, \Delta_1$, and $\Delta_2$ are mutually equivalent (Section 2). This characterization is applied to show that there exist operators that are *not* quasitriangular. On the other hand the set of quasitriangular operators is quite rich (Section 3); it is closed under the formation of polynomials, it is closed in the norm topology of operators, it is closed under the formation of countable direct sums, and it contains, for example, all operators of the form $N + K$ where $N$ is normal and $K$ is compact. The paper concludes with a few questions (Section 4). Sample: is it true for every operator $A$ that either $A$ or $A^*$ is quasitriangular?

## Section 1

Sequences of approximately invariant projections that are not required to be "large" always exist. A precise statement is this: for each operator $A$ there exists a sequence $\{E_n\}$ of non-zero projections of finite rank such that $\|AE_n - E_nAE_n\| \to 0$; in fact the $E_n$'s can be chosen to have rank 1. The proof is immediate from the existence of approximate eigenvalues and eigenvectors. Let $\lambda$ be a scalar and $\{e_n\}$ a sequence of unit vectors such that $\|Ae_n - \lambda e_n\| \to 0$. If the projections $E_n$ are defined by $E_nf = (f, e_n)e_n$, then

$$(AE_n - E_nAE_n)f = (f, e_n)(Ae_n - (Ae_n, e_n))e_n.$$

Since $(Ae_n, e_n) \to \lambda$, it follows that $\|AE_n - E_nAE_n\| \to 0$.

Since every operator has approximately invariant projections of rank 1, it is tempting to conclude, via the formation of finite spans, that every operator on an infinite-dimensional Hilbert space has approximately invariant projections of arbitrarily large finite ranks. The theory of approximate invariance turns out, however, to be surprisingly delicate. It is, for instance, not true that the span of two approximate eigenvectors is approximately invariant. More precisely, there exists a $3\times 3$ matrix $A$ and there exist two projections $F$ and $G$ of rank 1 such that $F$ is invariant under $A$, $G$ is nearly invariant under $A$, but if $E=F\vee G$, then $\|AE-EAE\|=1$. In detail: put

$$A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

let $F$ be the projection onto $\langle 0,1,0\rangle$, and let $G$ be the projection onto $\langle a,b,0\rangle$, where $|a|^2+|b|^2=1$ and $a$ is "small" (but not 0). It is easy to verify that

$$F=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad G=\begin{pmatrix} |a|^2 & ab^* & 0 \\ a^*b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad E=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\|AF-FAF\|=0,\quad \|AG-GAG\|=|a|,\quad \text{and}\quad \|AE-EAE\|=1.$$

This example, informal in its interpretation of "nearly invariant", can be used to construct an example of two sequences of approximately invariant projections, in the precise technical sense, such that the sequence of their spans is not approximately invariant, as follows. Let $H$ be the direct sum $H_1\oplus H_2\oplus\ldots$ of 3-dimensional spaces such as played a role in the preceding paragraph, and let the operator $A$ on $H$ be the direct sum $A_1\oplus A_2\oplus\ldots$ of the corresponding operators. Let $F_n$ be the direct sum projection whose summand with index $n$ is the previous $F$ and whose other summands are 0; let $G_n$ be the direct sum projection whose summand with index $n$ is the previous $G$ with $a=\frac{1}{n}$ and whose other summands are 0. It follows that $\|AF_n-F_nAF_n\|=0$ for all $n$, $\|AG_n-G_nAG_n\|\to 0$, and, if $E_n=F_n\vee G_n$, then $\|AE_n-E_nAE_n\|=1$.

It is slightly surprising that, despite the evidence of the preceding example, approximately invariant projections of arbitrarily large ranks always exist.

Theorem 1. *If $A$ is an operator on an infinite-dimensional Hilbert space, $\varepsilon$ is a positive number, and $n$ is a positive integer, then there exists a projection $E$ of rank $n$ such that $\|AE-EAE\|<\varepsilon$.*

Proof. For $n=1$, the result was derived from the existence of approximate eigenvectors. The idea of the inductive proof that follows is that although near invariance is not preserved by the formation of spans, it is preserved by the formation of orthogonal spans. Given $\varepsilon$ and $n$, assume the result for $n$, and let $F$ be a projection of rank $n$ such that $\|AF-FAF\|<\varepsilon/2$. Since the compression of $A$ to $\operatorname{ran}(1-F)$ (i.e., the restriction of $(1-F)A(1-F)$ to $\operatorname{ran}(1-F)$) has approximately invariant projections of rank 1, it follows that there exists a projection $G$ of rank 1 such that $G\perp F$ and

$$\|(1-F)A(1-F)G-G(1-F)A(1-F)G\|<\varepsilon/2.$$

(Find $G$ on $\operatorname{ran}(1-F)$ first and then extend it by defininining it to be 0 on $\operatorname{ran} F$.) Since $G(1-F)=(1-F)G=G$, the last inequality is equivalent to

$$\|(1-F)(1-G)AG\|<\varepsilon/2.$$

If $E=F+G$, then $E$ is a projection of rank $n+1$ and

$$\|AE-EAE\| = \|(1-E)AE\| = \|(1-F)(1-G)A(F+G)\| =$$

$$= \|(1-G)(1-F)AF+(1-F)(1-G)AG\| \leqq \|(1-F)AF\|+\|(1-F)(1-G)AG\|<\varepsilon.$$

## Section 2

It is trivial that the definition of quasitriangularity $(\Delta_1)$ implies the weakened form $(\Delta_2)$ (obtained from $(\Delta_1)$ by omitting the word "increasing"). It is also quite easy to prove that if, on a separable Hilbert space, $\liminf_{E\to 1}\|AE-EAE\|=0$ $(\Delta_0)$, then $A$ is quasitriangular $(\Delta_1)$. Indeed, let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for the space. By $(\Delta_0)$ there exists a projection $E_1$ of finite rank such that $e_1\in\operatorname{ran} E_1$ and $\|AE_1-E_1AE_1\|<1$. Again, by $(\Delta_0)$, there exists a projection $E_2$ of finite rank such that $E_1\leqq E_2$, $e_2\in\operatorname{ran} E_2$, and $\|AE_2-E_2AE_2\|<\frac{1}{2}$. In general, inductively, use $(\Delta_0)$ to get a projection $E_{n+1}$ of finite rank such that $E_n\leqq E_{n+1}$, $e_{n+1}\in\operatorname{ran} E_{n+1}$, and $\|AE_{n+1}-E_{n+1}AE_{n+1}\|<\dfrac{1}{n+1}$. Conclusion: $\{E_n\}$ is an increasing sequence of projections of finite rank such that $E_n\to 1$ and such that $\|AE_n-E_nAE_n\|\to 0$; in other words $A$ is quasitriangular, as promised.

The non-trivial implication along these lines is the one from $(\Delta_2)$ to $(\Delta_0)$. The proof depends on a lemma according to which if two projections have the same finite rank and are near, then there is a "small" unitary operator that transforms one onto the other. (For unitary operators "small" means "near to 1".) A possible quantitative formulation goes as follows.

Lemma 1. *If $E$ and $F$ are projections of the same finite rank such that $\|E-F\|=$ $=\varepsilon<1$, then the infimum of $\|1-W\|$, extended over all unitary operators $W$ such that $W^*EW=F$, is not more than $2\varepsilon^{\frac{1}{2}}$.*

The lemma can be improved, but the improvement takes considerably more work and for present purposes it is not needed. A trivial improvement is to drop the assumption that $E$ and $F$ have the same rank and recapture it from the known result [7, p. 58] that the inequality $\|E-F\|<1$ implies rank $E=$ rank $F$. Another qualitative improvement is to drop the assumption that the ranks are finite and pay for it by introducing partial isometries instead of unitary operators. The best kind of improvement is quantitative; the estimate $2\varepsilon^{\frac{1}{4}}$ can be sharpened to $2^{\frac{1}{2}}[1-(1-\varepsilon^2)^{\frac{1}{2}}]^{\frac{1}{2}}$. For a discussion of such results and references to related earlier work see [4]. Conjecturally the sharpened estimate is best possible, but the proof of that does not seem to be in the literature.

Proof. The equality of rank $E$ and rank $F$ implies the existence of a unitary operator $W_0$ such that $W_0^*EW_0=F$. Write $E$, $F$, and $W_0$ as operator matrices, according to the decomposition $1=E+(1-E)$, so that, for instance, $E=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. If $W_0=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, then $W_0^*=\begin{pmatrix}A^* & C^*\\ B^* & D^*\end{pmatrix}$, and therefore

$$W_0^*W_0=\begin{pmatrix}A^*A+C^*C & A^*B+C^*D\\ B^*A+D^*C & B^*B+D^*D\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The matrix of $F\,(=W_0^*EW_0)$ can now be computed; it turns out to be $\begin{pmatrix}A^*A & A^*B\\ B^*A & B^*B\end{pmatrix}$. Since the norm of each entry of a matrix is dominated by the norm of the matrix, it follows that

$$\|C^*C\|=\|1-A^*A\|\leqq\varepsilon \quad\text{and}\quad \|1-D^*D\|=\|B^*B\|\leqq\varepsilon.$$

Observe next that if $U$ and $V$ are unitary operators on ran $E$ and ran $(1-E)$ respectively, and if $W_1=\begin{pmatrix}U & 0\\ 0 & V\end{pmatrix}$, then $W_1$ commutes with $E$, and, therefore, $W_1W_0$ transforms $E$ onto $F$ (just as $W_0$ does). The purpose of the rest of the proof is to choose $U$ and $V$ so as to make $\|1-W_1W_0\|$ small. Since

$$1-W_1W_0=\begin{pmatrix}1-UA & -UB\\ -VC & 1-VD\end{pmatrix},$$

and since the norm of a matrix is dominated by the square root of the sum of the squares of the norms of its entries, it is sufficient to prove that by appropriate choices of $U$ and $V$ the entries of the last written matrix can be made to have small norms. The off-diagonal entries of $1-W_1W_0$ are easy to estimate:

$$\|-VC\|^2=\|C\|^2=\|C^*C\|\leqq\varepsilon \quad\text{and}\quad \|-UB\|^2=\|B\|^2=\|B^*B\|\leqq\varepsilon.$$

In these estimates $U$ and $V$ are arbitrary unitary operators; it is only in the next step that they have to be chosen so as to make something small.

Observe that since the ranks of $E$ and $F$ are finite, the lemma loses no generality if it is stated for finite-dimensional spaces only; the infinite-dimensional case is recaptured by applying the finite-dimensional lemma to $E$ and $F$ restricted to ran $E \vee$ ran $F$ and extending the resulting unitary operator by defining it to be the identity on the orthogonal complement of ran $E \vee$ ran $F$. In the finite-dimensional case $A$ is the product of a unitary operator and $(A^*A)^{\frac{1}{2}}$ (polar decomposition); let $U$ be the inverse of the unitary factor. With this choice $1-UA$ becomes $1-P$, where $P=(A^*A)^{\frac{1}{2}}$. Since $\|A\| \leqq 1$, so that $0 \leqq P^2 \leqq P \leqq 1$, it follows that $0 \leqq 1-P \leqq \leqq 1-P^2$, and hence that

$$\|1-UA\| = \|1-P\| \leqq \|1-P^2\| = \|1-A^*A\|.$$

A similar argument for $D$ produces a unitary $V$ such that $\|1-VD\| \leqq \|1-D^*D\|$. Conclusion:

$$\|1-W_1W_0\|^2 \leqq 2(\varepsilon+\varepsilon^2) \leqq 4\varepsilon,$$

and the proof of the lemma is complete.

The ground is now prepared for the proof of the principal result.

Theorem 2. *If $\{E_n\}$ is a sequence of projections of finite rank such that $E_n \to 1$ (strong topology) and such that $\|AE_n - E_nAE_n\| \to 0$, then* $\liminf_{E \to 1} \|AE - EAE\| = 0$.

Proof. It is to be proved that if $\varepsilon > 0$ and if $E_0$ is a projection with rank $E_0 = n_0 < \infty$, then there exists a projection $E$ of finite rank such that $E_0 \leqq E$ and $\|AE - EAE\| < \varepsilon$.

Let $\delta$ be a temporarily indeterminate positive number; it will be specified, in terms of $\varepsilon$, later. Suppose that $\{e_1, \ldots, e_{n_0}\}$ is an orthonormal basis for ran $E_0$. The two limiting assumptions imply the existence of a positive integer $n$ such that $\|e_j - E_ne_j\| < \delta/\sqrt{n_0}$ $(j=1, \ldots, n_0)$ and $\|AE_n - E_nAE_n\| < \delta$. The first of these inequalities implies that if $\delta$ is sufficiently small, then the set $\{E_ne_1, \ldots, E_ne_{n_0}\}$ is linearly independent. (The proof is easy and is omitted here; it is explicitly carried out in [6].) Let $F_0$ be the projection (of rank $n_0$) onto their span; note that $F_0 \leqq E_n$. (The $n$ here used will remain fixed from now on.)

If $f \in \operatorname{ran} E_0$, so that $f = \sum_{j=1}^{n_0} \xi_j e_j$, then

$$\|f - F_0f\|^2 = \|\sum_j \xi_j(e_j - E_ne_j)\|^2 \leqq \left(\sum_j |\xi_j| \cdot \|e_j - E_ne_j\|\right)^2 \leqq$$

$$\leqq \sum_j |\xi_j|^2 \cdot \sum_j \|e_j - E_ne_j\|^2 \leqq \|f\|^2 \cdot n_0\left(\delta/\sqrt{n_0}\right)^2,$$

and therefore

$$\|E_0 - F_0E_0\| \leqq \delta.$$

This shows that $E_0$ is approximately dominated by $F_0$; what is needed for the rest of the proof is the stronger assertion that $E_0$ is approximately equal to $F_0$.

By definition, ran $F_0$ is spanned by the vectors $F_0 e_j (=E_n e_j)$, $j=1, \ldots, n_0$; it follows that ran $F_0 =$ ran $F_0 E_0$. In other words, the restriction of $F_0$ to ran $E_0$ maps ran $E_0$ onto ran $F_0$. Call that restriction $T$; then $T$ is a linear transformation from a space of dimension $n_0$ onto a space of dimension $n_0$, and, consequently, $T$ is invertible. Since the spaces involved are finite-dimensional, the transformation $T^{-1}$ is bounded, but that is not enough information; what is needed is an effective estimate of $\|T^{-1}\|$. That turns out to be easy to get. If $f \in \operatorname{ran} E_0$, then

$$\|F_0 f\| \geqq \|f\| - \|f - F_0 f\| \geqq \|f\| - \delta\|f\| = (1-\delta)\|f\|,$$

and therefore $\|T^{-1}\| \leqq \dfrac{1}{1-\delta}$.

The inequality $\|E_0 - F_0 E_0\| \leqq \delta$ shows that $F_0$ is near to $E_0$ on ran $E_0$; the next step is to show that $F_0$ is near to $E_0$ on $\operatorname{ran}^{\perp} E_0$. Suppose therefore that $f \perp \operatorname{ran} E_0$, i.e., that $E_0 f = 0$, and write $g = T^{-1} F_0 f$. Since $g \in \operatorname{ran} E_0$, it follows that $F_0 g = Tg = F_0 f$, or $F_0 E_0 g = F_0 f$; note that $\|g\| \leqq \dfrac{1}{1-\delta}\|f\|$. Since $\|F_0 f - E_0 g\| \leqq$ $\leqq \|F_0 f - F_0 E_0 g\| + \|F_0 E_0 g - E_0 g\| \leqq \delta\|g\| \leqq \dfrac{\delta}{1-\delta}\|f\|$, it follows that

$$\|F_0 f\|^2 = (F_0 f, f) \leqq |(F_0 f - E_0 g, f)| + |(E_0 g, f)| \leqq \frac{\delta}{1-\delta}\|f\|^2$$

($(E_0 g, f) = 0$ because $E_0 f = 0$), and hence that

$$\|F_0(1-E_0)\| \leqq \left(\frac{\delta}{1-\delta}\right)^{1/2}.$$

This inequality together with $\|E_0 - F_0 E_0\| \leqq \delta$ yields

$$\|E_0 - F_0\| \leqq \delta + \left(\frac{\delta}{1-\delta}\right)^{1/2} = \gamma.$$

Lemma 1 is now applicable. Choose $\delta$ small enough to make sure that $\gamma < 1$ and conclude that there exists a unitary operator $W$ such that $W^* E_0 W = F_0$ and $\|1 - W\| \leqq 2\sqrt{\gamma}$. Write $E = W E_n W^*$. Since $F_0 \leqq E_n$, it follows that $E_0 \leqq E$; all that remains is to verify that $E$ can be forced to be within $\varepsilon$ of being invariant under $A$. That is easy; since

$$\|AE - EAE\| = \|A(WE_nW^*) - (WE_nW^*)A(WE_nW^*)\|,$$

and since the right hand term depends continuously on $W$, it follows that if $W$ is chosen sufficiently near to 1 (i.e., if $\delta$ is chosen sufficiently small), then the right hand term can be made arbitrarily near to $\|AE_n - E_nAE_n\|$, within $\varepsilon/2$ of it, say. Since $\|AE_n - E_nAE_n\| < \delta$, it might now be necessary to make $\delta$ a little smaller still, so as to guarantee $\delta < \varepsilon/2$; after this modification it will follow that, indeed, $\|AE - EAE\| < \varepsilon$.

The first definition of quasitriangularity ($\Delta_1$) is quite hard ever to disprove; how does one show that there does not exist a sequence with the required properties? Theorem 2 makes the job easier. For an example suppose that $\{e_0, e_1, e_2, \ldots\}$ is an orthonormal basis and let $U$ be the corresponding unilateral shift. The properties of $U$ that will be needed are that it is an isometry ($U^*U = 1$) whose adjoint has a non-trivial kernel ($U^*e_0 = 0$).

Theorem 3. *The unilateral shift is not quasitriangular.*

Proof. Let $E_0$ be the projection (of rank 1) onto $e_0$. The proof will show that if $E$ is a projection of finite rank such that $E_0 \leqq E$ (i.e., $e_0 \in \operatorname{ran} E$), then $\|UE - EUE\| = 1$.

Put $D = UE - EUE = (1-E)UE$. Clearly $\|D\| \leqq 1$; the problem is to prove the reverse inequality. Observe that $D^*D = EU^*(1-E)\cdot(1-E)UE = EU^*UE - EU^*EUE = E - (EUE)^*(EUE)$. The finite-dimensional space $\operatorname{ran} E$ reduces both $E$ and $EUE$, and on its orthogonal complement both those operators vanish. It follows that if $T$ is the restriction of $EUE$ to $\operatorname{ran} E$, then $\|D^*D\| = \|1 - T^*T\|$; the symbol "1" here refers, of course, to the identity operator on $\operatorname{ran} E$.

Now use the assumption that $e_0 \in \operatorname{ran} E$ and observe that $T^*e_0 = EU^*Ee_0 = EU^*e_0 = 0$. Since $T^*$ is an operator on a finite-dimensional space and has a non-trivial kernel, the same is true of $T^*T$. (The falsity of this implication on infinite-dimensional spaces is shown by $U$ itself.) If $f$ is a unit vector in $\ker T^*T$, then $\|(1 - T^*T)f\| = 1$, and therefore $\|1 - T^*T\| \geqq 1$; the proof of the theorem is complete.

## Section 3

It is not difficult to see that a polynomial in a quasitriangular operator is quasitriangular. Suppose indeed that $\{E_n\}$ is a sequence of projections such that $\|AE_n - E_nAE_n\| \to 0$, and let $p$ be a polynomial. Since $AE_n - E_nAE_n$ is linear in $A$, it is sufficient to prove the assertion for monomials, $p(z) = z^k$, and that can be done by induction. The case $k=1$ is covered by the hypothesis. (Note icidentally that constant terms can come and go with impunity: $(A+\lambda)E_n - E_n(A+\lambda)E_n = AE_n - E_nAE_n$.) The induction step from $k$ to $k+1$ is implied by the identity:

$$(1-E_n)A^{k+1}E_n = \big((1-E_n)A^{k+1}E_n - (1-E_n)AE_nA^kE_n\big) + (1-E_n)AE_nA^kE_n =$$
$$= (1-E_n)A\big((1-E_n)A^kE_n\big) + \big((1-E_n)AE_n\big)A^kE_n.$$

W. B. ARVESON has proved that an operator similar to a quasitriangular one is also quasitriangular. The result of the preceding paragraph and ARVESON's result are closure properties of the set of all quasitriangular operators. The next two results are of the same kind.

Theorem 4. *A countable direct sum of quasitriangular operators is quasitriangular.*

Proof. Suppose that for each $j(=1, 2, 3, \ldots)$ $A^{(j)}$ is an operator and $\{E_n^{(j)}\}$ is a sequence of projections of finite rank such that $\|A^{(j)}E_n^{(j)} - E_n^{(j)}A^{(j)}E_n^{(j)}\| \to 0$ as $n \to \infty$. Write $A = A^{(1)} \oplus A^{(2)} \oplus \ldots$. For each fixed $k$, find $n_k$ so that $\|A^{(j)}E_n^{(j)} - E_n^{(j)}A^{(j)}E_n^{(j)}\| < \frac{1}{k}$ when $1 \leqq j \leqq k$ and $n \geqq n_k$; write $E_k = E_{n_k}^{(1)} \oplus \ldots \oplus E_{n_k}^{(k)} \oplus 0 \oplus 0 \oplus \ldots$. The $E_k$'s are projections of finite rank. Since $E_{n_k}^{(j)} \to 1$ as $k \to \infty$ (strong topology) for each $j$, it follows that

$$E_k\langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle \to \langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle$$

whenever the vector $\langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle$ is finitely non-zero. The boundedness of the sequence $\{E_k\}$ implies that $E_k \to 1$ (strong topology). Since $\|AE_k - E_kAE_k\| = \max\{\|A^{(j)}E_{n_k}^{(j)} - E_{n_k}^{(j)}A^{(j)}E_{n_k}^{(j)}\|:\ j = 1, \ldots, k\} < \frac{1}{k}$, the proof is complete.

Theorem 5. *The set of quasitriangular operators is closed in the norm topology.*

Proof. Suppose that $A_n$ is quasitriangular and $\|A_n - A\| \to 0$. Given a positive number $\varepsilon$ and a projection $E_0$ of finite rank, find $n_0$ so that $\|A - A_{n_0}\| < \varepsilon/3$, and then find a projection $E$ of finite rank such that $E_0 \leqq E$ and $\|A_{n_0}E - EA_{n_0}E\| < \varepsilon/3$. It follows that $\|AE - EAE\| \leqq \|AE - A_{n_0}E\| + \|A_{n_0}E - EA_{n_0}E\| + \|EA_{n_0}E - EAE\| < \varepsilon$.

Theorem 4 implies (and it is obvious anyway) that (on a separable Hilbert space, as always) every diagonal operator is quasitriangular. Since every normal operator is in the closure of the set of diagonal operators, Theorem 5 implies that every normal operator is quasitriangular.

A similar application of Theorem 5 shows that every compact operator is quasitriangular; what is needed is the easy observation that every operator of finite rank is quasitriangular. For compact operators, however, more is true; not only does there exist a well behaved sequence of projections, but in fact all "large" sequences are well behaved. That is: if $A$ is compact and if $\{E_n\}$ is a sequence of projections such that $E_n \to 1$ (strong topology), then $\|AE_n - E_nAE_n\| \to 0$. The following formulation in terms of the directed set of projections of finite rank is more elegant; the assertion is that lim inf can be replaced by lim.

Lemma 2. *If $A$ is compact, then* $\lim_{E \to 1} \|AE - EAE\| = 0$.

Proof. Given a positive number $\varepsilon$, find an operator $F$ of finite rank such that $\|A-F\|<\varepsilon/2$, and then find a projection $E_0$ of finite rank such that $FE_0=E_0F=F$. If $E$ is a projection of finite rank such that $E_0 \leqq E$, then

$$\|AE-EAE\| \leqq \|AE-FE\| + \|FE-EFE\| + \|EFE-EAE\| < \varepsilon.$$

Lemma 2 implies that an operator of the form $A+K$, where $A$ is quasitriangular and $K$ is compact, is quasitriangular; in particular so is every operator of the form $N+K$, where $N$ is normal and $K$ is compact.

Still other quasitriangular operators of interest have arisen in the various generalizations of the Aronszajn—Smith theorem on invariant subspaces of compact operators. Thus, for instance, a crucial step in the treatment of polynomially compact operators [5] is the proof that every polynomially compact operator with a cyclic vector is quasitriangular. In their generalization of the invariant subspace theorem for polynomially compact operators, ARVESON and FELDMAN [2] need and prove the statement that every quasinilpotent operator with a cyclic vector is quasitriangular.

## Section 4

Quasitriangular operators first arose in connection with the invariant subspace problem, but their status in that connection is still not settled.

Question 1. *Does every quasitriangular operator have a non-trivial invariant subspace?*

Experience with compact and polynomially compact operators suggests that the answer to Question 1 is yes. On the other hand, if the answer is yes, then it follows that every quasinilpotent operator has a non-trivial invariant subspace. Since it is a not unreasonable guess that the general invariant subspace question is equivalent to the one for quasinilpotent operators, and since the answer to the general invariant subspace question is more likely no than yes, the compact and polynomially compact experience comes under suspicion.

PETER ROSENTHAL suggested a more concrete way of connecting Question 1 with quasinilpotent operators. It is quite a reasonable conjecture that the spectrum of every unicellular operator is a singleton. (An operator is unicellular if its lattice of invariant subspaces is a chain.) Every transitive operator is obviously unicellular. (An operator is transitive if it has no non-trivial invariant subspaces.) The truth of the conjecture would imply therefore that, except for an additive scalar, every transitive operator is quasinilpotent, and hence, once again, an affirmative answer to Question 1 would imply an affirmative answer to the general invariant subspace question.

Question 2. *If the direct sum of two operators is quasitriangular, are both summands quasitriangular?*

This question is due to Carl Pearcy. He has proved that if $A\oplus 0$ is quasitriangular, then $A$ must be, but the general case is open. An interesting related question concerns the unilateral shift $U$: is $U\oplus U^*$ quasitriangular? If the answer to Question 2 is yes, then the answer to this question about $U$ must be no. What is known, as a special case of Pearcy's result, is that $U\oplus 0$ is not quasitriangular.

Question 3. *Is it true for every operator that either it or its adjoint is quasitriangular?*

The only example presented above of an operator that is not quasitriangular is the unilateral shift $U$; a glance at the matrix of $U$ proves that $U^*$ is quasitriangular. If the answer to Question 3 is yes, then Question 1 is equivalent to the general invariant subspace question. Since $U\oplus U^*$ is unitarily equivalent to its own adjoint, it follows that an affirmative answer to Question 3 would imply that $U\oplus U^*$ is quasitriangular, and, therefore, that the answer to Question 2 is no. There are other interesting and unknown special cases of Question 3. Thus, for instance, by an improvement of the argument that proved that $U$ is not quasitriangular, Pearcy has obtained a large class of operators that are not quasitriangular; one of them is $3U+U^*$. It is not known whether the adjoint of that operator is quasitriangular.

## References

[1] N. Aronszajn and K. T. Smith, Invariant subspaces of completely continuous operators, *Ann. of Math.*, **60** (1954), 345—350.

[2] W. B. Arveson and J. Feldman, A note on invariant subspaces, *Mich. Math. J.*, **15** (1968), 61—64.

[3] A. R. Bernstein and A. Robinson, Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos, *Pac. J. Math.*, **16** (1966), 421—341.

[4] H. O. Cordes, On a class of $C^*$-algebras, *Math. Ann.*, **170** (1967), 283—313.

[5] P. R. Halmos, Invariant subspaces of polynomially compact operators, *Pac. J. Math.*, **16** (1966), 433—437.

[6] P. R. Halmos, Irreducible operators, *Mich. Math. J.*, **15** (1968), 215—223.

[7] B. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes* (Berlin, 1942).

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