A characterization of thin operators

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Let $\mathfrak{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded, linear operators on $\mathfrak{H}$. In [2], HALMOS initiated the study of the class of *quasitriangular* operators on $\mathfrak{H}$. These operators may be defined as follows. Let $\mathcal{P}$ denote the directed set consisting of all finite dimensional (orthogonal) projections in $\mathcal{L}(\mathfrak{H})$ under the usual ordering ($P \leqq Q$ if and only if $(Px, x) \leqq (Qx, x)$ for all $x \in \mathfrak{H}$.) For a fixed $A \in \mathcal{L}(\mathfrak{H})$, the map $P \to \|PAP - AP\|$ is a net on $\mathcal{P}$, and A is quasitriangular provided

$$\liminf_{P \in \mathcal{P}} \|PAP - AP\| = 0.$$

(The definition of quasitriangularity given in [2] is actually somewhat different. That the above is an equivalent definition is [2, Theorem 2].) Among the quasi-triangular operators are the operators of the form $\lambda + C$ where λ is a scalar and C is a compact operator. In this note we call such operators $\lambda + C$ *thin* operators. Among the quasitriangular operators are also the operators A that satisfy

(H) $$\lim_{P \in \mathcal{P}} \|PAP - AP\| = 0.$$

HALMOS has conjectured that an operator has property (H) if and only if it is thin. The purpose of this note to prove that conjecture.

To accomplish this, we first obtain an interesting characterization of the η-function of BROWN and PEARCY [1] in terms of the nets $P \to \|PAP - AP\|$. Recall that the η-function is defined on $\mathcal{L}(\mathfrak{H})$ by the equation

$$\eta(A) = \lim_{P \in \mathcal{P}} \Big[\sup_{\substack{x \in (1-P)\mathfrak{H} \\ \|x\| = 1}} \|Ax - (Ax, x)x\| \Big].$$

Theorem 1. *For every* $A \in \mathcal{L}(\mathfrak{H})$,

$$\eta(A^*) = \limsup_{P \in \mathcal{P}} \|PAP - AP\|.$$

Proof. It clearly suffices to prove that

$$\eta(A) = \limsup_{P \in \mathcal{P}} \|PA(1-P)\|$$

for every $A\in\mathscr{L}(\mathfrak{H})$, since then

$$\eta(A^*)=\limsup_{P\in\mathscr{P}}\|PA^*(1-P)\|=\limsup_{P\in\mathscr{P}}\|(1-P)AP\|.$$

Thus let $A\in\mathscr{L}(\mathfrak{H})$ be fixed, and let

$$\limsup_{P\in\mathscr{P}}\|PA(1-P)\|=\alpha.$$

Let also $\varepsilon>0$ and $P_0\in\mathscr{P}$ be given. Then, by definition, there exists $P_1\in\mathscr{P}$ such that $P_1\geqq P_0$ and $\|P_1A(1-P_1)\|>\alpha-\varepsilon$. It follows that there is a unit vector $y\in(1-P_1)\mathfrak{H}$ such that $\|P_1A(1-P_1)y\|>\alpha-\varepsilon$. Since Ay can be written as $Ay=$ $=[Ay-(Ay,y)y]+(Ay,y)y$ and $P_1y=0$, we have

$$\|Ay-(Ay,y)y\|\geqq\|P_1Ay\|=\|P_1A(1-P_1)y\|>\alpha-\varepsilon.$$

Since $\eta(A)$ can be written as

$$\eta(A)=\limsup_{P\in\mathscr{P}}[\sup_{\substack{x\in(1-P)\mathfrak{H}\\ \|x\|=1}}\|Ax-(Ax,x)x\|],$$

we have shown that $\eta(A)\geqq\alpha$.

To complete the proof, we show that $\alpha\geqq\eta(A)$. Let $\delta>0$ and a finite dimensional projection P_2 be given. It suffices to exhibit a finite dimensional projection $Q\geqq P_2$ and a unit vector z in the range of $1-Q$ such that $\|QA(1-Q)z\|>\eta(A)-\delta$. To find such a projection Q and such a vector z, we proceed as follows. The definition of $\eta(A)$ guarantees that there exists a projection $P_3\in\mathscr{P}$ such that for every finite dimensional projection $P\geqq P_3$, there exists a unit vector x_p in the range of $(1-P)$ such that $\|Ax_p-(Ax_p,x_p)x_p\|>\eta(A)-\delta$. Choose $P_4\geqq P_2, P_3$, and let $z(=x_{p_4})$ be a unit vector in the range of $(1-P_4)$ such that $\|Az-(Az,z)z\|>\eta(A)-\delta$. Finally, let Q be the finite dimensional projection that is the supremum of P_4 and the one dimensional projection whose range is $Az-(Az,z)z$. Since z is perpendicular to the range of P_4 and also to the vector $Az-(Az,z)z$, z is perpendicular to the range of Q. In other words, z is a unit vector in the range of $1-Q$, and the inequality

$$\|QA(1-Q)z\|=\|QAz\|=\|Q[Az-(Az,z)z]+Q(Az,z)z\|=\|Az-(Az,z)z\|>\eta(A)-\delta$$

completes the proof.

Theorem 2. *An operator $A\in\mathscr{L}(\mathfrak{H})$ has property (H) if and only if A is thin.*

Proof. Clearly A is thin if and only if A^* is thin, and according to [1, Theorem 1], A^* is thin if and only if $\eta(A^*)=0$. Finally, from Theorem 1 we see that $\eta(A^*)=0$ if and only if

$$\limsup_{P\in\mathscr{P}}\|(1-P)AP\|=0,$$

or, what is the same thing, if and only if A has property (H).

We conclude this note by observing that the problem treated above makes sense in any von Neumann algebra. To be specific, let $\mathscr{A}$ be any von Neumann algebra, let $\mathscr{J}$ be any uniformly closed ideal in $\mathscr{A}$, and let $\mathscr{P}$ denote the directed set of projections in $\mathscr{J}$. It is not hard to see that every operator of the form $A=\lambda+J$, where $J\in\mathscr{J}$, satisfies

$$\lim_{P\in\mathscr{P}}\|PAP-AP\|=0.$$

Is the converse true?

Bibliography

[1] A. Brown and C. Pearcy, Structure of commutators of operators, *Ann. of Math.*, **82** (1965), 112—127.
[2] P. R. Halmos, Quasitriangular operators, *Acta Sci. Math.*, **29** (1968), 283—293.

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(Received January 8, 1968)