A remark on a class of power-bounded operators in Hilbert space

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The notion of unitary dilation was generalized by SZ.-NAGY and C. FOIAŞ [1], [2] by considering the classes C_ϱ of operators T in Hilbert spaces H whose powers admit a representation

$$T^n = \varrho \cdot \mathrm{pr}\ U^n \qquad (n = 1, 2, 3, \ldots)$$

where U is a unitary operator in some Hilbert space $H_1 \supset H$.

H. LANGER has proposed the following further generalization: if A is a positive self-adjoint operator, $mI \leqq A \leqq MI$, where $m > 0$, consider the class C_A of operators whose powers admit a representation

$$QT^nQ = \mathrm{pr}\ U^n \qquad (n = 1, 2, 3, \ldots)$$

where $Q = A^{-\frac{1}{2}}$ and U is a unitary operator in some Hilbert space $H_1 \supset H$; see [2], p. 54.

The aim of this note is to prove the following

Theorem. *C_A is a increasing function of A in the sense that $A_1 \leqq A_2$ implies $C_{A_1} \subseteq C_{A_2}$.*

Proof. We use the following characterization of C_A indicated by H. LANGER (see [2], p. 54): $T \in C_A$ if and only if

1° the spectrum of T lies in the closed unit disc,

2° $(Ah, h) - \mathrm{Re}\,(z(A-I)Th, h) + |z|^2((A-2I)Th, Th) \geqq 0$ for $|z| \leqq 1$ and $h \in H$.

The relation 2° can be written in the form:

$$((A-I)h, h) + (h, h) - 2\mathrm{Re}\,(z(A-I)Th, h) - |z|^2\|Th\|^2 + |z|^2((A-I)Th, Th) \geqq 0$$

or, equivalently,

$$\|h\|^2 - |z|^2\|Th\|^2 + ((A-I)(I-2T)h, (I-zT)h) \geqq 0.$$

Since the left-hand side is an increasing function of the self-adjoint operator A, the theorem is proved.

Corollary 1. *If $T \in C_A$ then $T \in C_{\|A\|}$.*
This follows from the fact that $A \leqq \|A\| \cdot I$.

Corollary 2. *Every operator T in C_A is similar to a contraction.*
This follows from Corollary 1 and the theorem of [3].

Corollary 3. *There exist power-bounded operators which do not belong to any class C_A.*

Indeed in [1] there is given a power-bounded operator which belongs to none of the classes C_ϱ, thus it belongs to none of the classes C_A, either.

References

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[3] Similitude des opérateurs de classe C_ϱ à des contractions, *C. R. Acad. Sci. Paris*, **264** (1967), 1063—1065.

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