A problem on lacunary series

By R. KAUFMAN in Urbana (Illinois, U.S.A.)

The limiting distributions of lacunary trigonometric sums

(1) $$S_N(t) = (\tfrac{1}{2}N)^{-\frac{1}{2}} \sum_{k=1}^{N} \cos(y_k t)$$

where $1 \leqq y_1$, $qy_k \leqq y_{k+1}$ $(1 \leqq k < \infty)$, were considered by SALEM and ZYGMUND [3]. who showed that, over any fixed set of positive Lebesgue measure, S_N tends in law to the normal, or Gaussian, distribution of mean 0 and variance 1. HELSON and KAHANE [1] showed that certain consequences of lacunarity persist if the Lebesgue measure is replaced by a Baire probability measure whose Fourier—Stieltjes transform meets the condition $\hat{\mu}(u) = O(|u|^{-\alpha})$ for some $\alpha > 0$. The nearest metric analogue of this is

(2) $$\mu([a, a+h]) \leqq Mh^{\beta} \quad \text{for all } a, \text{ and } h > 0,$$

where β and $M = M(\beta)$ are positive constants. In this case nothing like the Salem—Zygmund result is necessarily valid, even if (2) holds for every $\beta < 1$. However, if we treat the coefficients y_k as functions of a variable x, we can obtain a similar result, at least in a special case:

Theorem. *For $x > 1$, let $y_k = x^k$, that is*

$$S_N(t) = (\tfrac{1}{2}N)^{-\frac{1}{2}} \sum_{k=1}^{N} \cos(x^k t).$$

Then for almost all $x > 2$

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-i\lambda S_N(t)} \mu(dt) = e^{-\frac{1}{2}\lambda^2} \qquad (-\infty < \lambda < \infty).$$

1. Lemma. *There exist a number $\delta > 0$, depending only on β, and a number M' depending only on M and β (cf. (2)) with the property: For any real function $f(x)$, $0 \leqq x \leqq 1$, of class C^2, with $f'' \geqq r > 0$,*

$$I = \int_0^1 \left| \int_{-\infty}^{\infty} e^{itf(x)} \mu(dt) \right|^2 dx \leqq M' r^{-\delta}.$$

Proof. The inner integral can be written

$$\int_{-\infty} \int_{-\infty} e^{i(t-s)f(x)} \mu(dt)\,\mu(ds)$$

so

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} e^{i(t-s)f(x)}\,dx\,\mu(dt)\,\mu(ds) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-s)\,\mu(dt)\,\mu(ds).$$

By VAN DER CORPUT's lemma [2]

$$|h(t-s)| \leqq C|r(t-s)|^{-\frac{1}{2}} \qquad (-\infty < s,\ t < \infty),$$

for an absolute constant C; clearly also $|h| \leqq 1$. Let $\eta > 0$ so that

$$\iint_{|t-s| \leqq \eta} \mu(dt)\,\mu(s) = \int_{-\infty}^{\infty} \int_{s-\eta}^{s+\eta} \mu(dt)\,\mu(ds) \leqq 2M\eta^{\beta}.$$

For every choice of $\eta > 0$ we find

$$I \leqq 2M\eta^{\beta} + Cr^{-\frac{1}{2}}\eta^{-\frac{1}{2}}.$$

Choosing $\eta = r^{-1/1+2\beta}$ we obtain

$$I \leqq (2M + C)\,r^{-\beta/1+2\beta}.$$

2. If $c_1, \ldots, c_N$ are real numbers, then

$$\left| \exp\left[-i \sum_{k=1}^{N} c_k\right] - \sum_{k=1}^{N} (1 - ic_k) \sum_{k=1}^{N} \left(1 - \frac{1}{2} c_k^2\right) \right| \leqq 2 \sum_{k=1}^{N} |c_k|^3 \sum_{k=1}^{N} (1 + c_k^2)^{3/2}.$$

Hence

$$\exp(-i\lambda S_N(t)) = O(N^{-\frac{1}{2}}) + \sum_{k=1}^{N} \left(1 - i\lambda(\tfrac{1}{2}N)^{-1} \cos(x^k t)\right) \sum_{k=1}^{N} \left(1 - \lambda^2 N^{-1} \cos^2(x^k t)\right).$$

(The symbols O and o always refer to a bound, uniform for any interval $-B \leqq \lambda \leqq B$). Using the formula $\cos 2v = 2\cos^2 v - 1$, we see that the second factor converges in μ-measure to $e^{-\frac{1}{2}\lambda^2}$, provided

$$\sum_{1 \leqq j < k \leqq N} \sum \left| \int_{-\infty}^{\infty} \cos(2x^j t) \cos(2x^k t)\,\mu(dt) \right| = o(N^2),$$

or

$$\sum_{1 \leqq j < k \leqq N} \sum \left| \int_{-\infty}^{\infty} \cos(2x^j t \pm 2x^k t)\,\mu(dt) \right| = o(N^2).$$

But by the lemma and the Schwarz inequality

$$\sum_{1 \leqq j < k} \sum \int_{q}^{q+1} \left| \int_{-\infty}^{\infty} \cos(2x^j t \pm 2x^k t)\,\mu(dt) \right| dx < \infty,$$

for any $q>1$. Here we applied the lemma to the functions $2x^k \pm 2x^j$ $(1 \leqq j<k)$ whose derivatives are easily estimated. It follows that the convergence of the second factor takes place for almost all $x>1$.

3. Set

$$J_N(x) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} [1 - i\lambda(\tfrac{1}{2}N)^{-\frac{1}{2}} \cos(x^k t)]\mu(dt).$$

First, we estimate the sum of the terms involving a single cosine, or a product of two cosines. The first kind give a sum

$$\prod_{k=1}^{N} O(N^{-\frac{1}{2}}) \int_{-\infty}^{\infty} \cos(x^k t)\,\mu(dt)$$

and the second

$$\sum_{1 \leqq j<k \leqq N}\sum O(N^{-1}) \int_{-\infty}^{\infty} \cos(x^k t \pm x^j t)\,\mu(dt).$$

We already dealt with integrals similar to these, and showed that the sums converge to 0 for almost all $x>1$.

From now on we assume

$$q+1 \geqq x \geqq q>2$$

so that if $0<k_1<\ldots<k_r$ are integers $(2 \leqq r)$,

$$\frac{d^2}{dx^2}(x^{k_r} \pm x^{k_{r-1}} \pm \ldots \pm x^{k_1}) \geqq Aq^{k_r}$$

for some $A=A(q)>0$.

Consider now the part of $J_N(x)$ involving products of exactly m cosines, $3 \leqq m \leqq N$. We divide this further according to the largest power of x involved, and obtain

$$\sum_{1 \leqq k_1<\ldots<k_m \leqq N} \cdots \sum \left(\frac{\sqrt{2}}{2}\right)^m O(B^m N^{-\frac{1}{2}m}) \int_{-\infty}^{\infty} \cos(\pm x^{k_m} t \pm \ldots \pm x^{k_1} t)\,\mu(dt).$$

(The number B is chosen so that $|\lambda| \leqq B$.)

By the lemma and the inequality on the second derivative, there is an $r \in (0, 1)$ such that

$$\int_q^{q+1} \left| \int_{-\infty}^{\infty} \cos(tx^{k_m} \pm \ldots \pm tx^{k_1})\,\mu(dt) \right| dx = O(r^{k_m}).$$

Thus the integral of the modulus of that part of $J_N(x)$ not already disposed of, is of the order of magnitude

$$\varphi(N) = \sum_{m=3}^{N} N^{-\frac{1}{2}m}(2B)^m \sum_{k=m}^{N} \binom{k-1}{m-1} r^k \leqq \sum_{k=3}^{N} \sum_{m=3}^{k} r^k N^{-\frac{1}{2}m}(2B)^m \binom{k}{m}.$$

By TAYLOR's formula

$$\sum_{m=3}^{k}(2BN^{-\frac{1}{2}})^m \binom{k}{m} \leqq \frac{1}{6}(2BN^{-\frac{1}{2}})^3 \sum_{m=3}^{k}(2BN^{-\frac{1}{2}})^{m-3}\binom{k}{m} m(m-1)(m-2) \leqq$$
$$\leqq k^3(2BN^{-\frac{1}{2}})^3(1+2BN^{-\frac{1}{2}})^k.$$

Thus $\varphi(N) \ll \sum_{k=3}^{N} N^{-\frac{3}{2}} r^k k^3 (1+2BN^{-\frac{1}{2}})^k$. If N is so large that $(1+2BN^{-\frac{1}{2}})\,r < r^{\frac{1}{2}}$, $\varphi(N) \ll N^{-\frac{3}{2}}$. Since $\sum N^{-\frac{3}{2}} < \infty$, $J_N(x) \to 0$ for almost all $x \in [q, q+1]$, and the proof is complete.

References

[1] H. HELSON and J.-P. KAHANE, A Fourier method in diophantine problems, *J. Analyse Math.*, **15** (1965), 245—262.

[2] R. KERSHNER, Determination of a ... constant, *American J. Math.*, **57** (1935), 840—846.

[3] R. SALEM and A. ZYGMUND, On lacunary trigonometric series, *Proc. Nat. Acad. Sci.*, **33** (1947), 333—338, and **34** (1948), 54—62.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

(Received November 1, 1967)