On general multiplication of infinite series

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The most general definition of the product of two infinite series can be obtained as follows:

Definition 1. Let us denote by N the set of all pairs (k, l) of positive integers, and let π be a sequence $N_1, N_2, \ldots, N_m, \ldots$ of finite, mutually disjoint subsets of N such that $N = \bigcup_{m=1}^{\infty} N_m$. Given two infinite series [1])

$$A = \sum_{k=1}^{\infty} a_k \quad \text{and} \quad B = \sum_{l=1}^{\infty} b_l \tag{1.1}$$

we call the series

$$C = \sum_{m=1}^{\infty} c_m = \sum_{m=1}^{\infty} \Big(\sum_{(k,l) \in N_m} a_k b_l \Big) \tag{1.2}$$

the product of the series (1.1) *obtained by the method corresponding to the sequence* π, or simply *by the method* (π), and we denote it by $\pi(\Sigma a_k, \Sigma b_l)$ or shortly by $\pi(A, B)$.

Definition 2. The method (π) will be called *perfect* if for any two convergent series (1. 1), the product series $\pi(A, B)$ also converges and its sum is equal to the product of the sums of the factor series.

Definition 3. The method (π) will be said to have property M_1 (resp. M_2) if for any series A and B the convergence of A (resp. B) and the absolute convergence of B (resp. A) implies the convergence of $\pi(A, B)$, its sum being equal to the product of the sums of the factor series.

Definition 4. If a method (π) has both properties M_1 and M_2 we will say that it has the *Mertens property.*

[1]) In the sequel the series Σa_k, Σb_l, Σc_m, ... will be denoted by the corresponding capital letters $A, B, C, \ldots$ independently of their convergence or divergence.

Definition 5. The method (π) will be said to have the *Abel property*, if for any convergent series A and B the convergence of $\pi(A, B)$ implies that the sum of $\pi(A, B)$ is equal to the product of the sums of A and B.

R. RADO [1] has given necessary and sufficient conditions for a method (π) to be perfect. RADO considers merely methods with sets N_m consisting of one element. For the general case of definition 1 the perfectness and the Mertens property were characterized by A. ALEXIEWICZ [2] in necessary and sufficient form.

Although the theorems of ALEXIEWICZ solve the convergence problem of a method (π) — apart from the Abel property — in the classical sense, there arises the following question:

If we pass over the classical view of convergence, i. e. if we agree that the "convergence" of a series Σc_n means that the series $\Sigma \pm c_n$ is convergent with the probability 1 taking at random the signs of its terms, then how can we modify the theorems of ALEXIEWICZ, and how we stand with the problem of the Abel property?

We can formulate our problem — due to RADEMACHER [3], KOLMOGOROFF and KHINTCHINE [4] — analytically in the following manner:

If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of the Rademacher functions i.e. if

$$r_n(t) = \operatorname{sign}(\sin 2^n \pi t) \qquad (n=1, 2, 3, \ldots) \tag{1.3}$$

in the interval $0 \leqq t \leqq 1$, then for a given method (π) what can we say about the convergence of the series

$$\pi(A(x), B(y)) = \sum_{m=1}^{\infty} \Big(\sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \Big) \tag{1.4}$$

at the points (x, y) of the unit square $Q = \{(x, y); 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$, assuming that the factor series

$$A(x) = \sum_{k=1}^{\infty} a_k r_k(x) \quad \text{and} \quad B(y) = \sum_{l=1}^{\infty} b_l r_l(y) \tag{1.5}$$

are convergent almost everywhere in [0, 1], i.e. assuming that the conditions $\Sigma a_k^2 < \infty$ and $\Sigma b_l^2 < \infty$ are fulfilled?

In section 2 we shall prove that every method (π) possesses the Mertens property in the above sense.

In section 3 we shall show that every method (π) becomes perfect if we put the terms of the product series into brackets in suitable form, and at the same time we mention a conjecture in the theory of Walsh series, which is essentially equivalent to the perfectness of every method (π).

Finally in section 4 we prove that every method (π) has the Abel property.

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Theorem 1. *If the conditions*

$$\sum_{k=1}^{\infty} |a_k| < \infty \quad \textit{and} \quad \sum_{l=1}^{\infty} b_l^2 < \infty \tag{2.1}$$

hold, then the product series (1. 4) *of the series* (1. 5) — *generated by an arbitrary given method* (π) — *converges almost everywhere on the unit square* Q.

Proof. First of all we cite a lemma — discovered by ZYGMUND and MARCINKIEWICZ [5] — which will be used in the sequel.

Lemma 1. *If the functions of an orthonormal system* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *in* $L^2(0, 1)$ *are stochastically independent*[1]) *with the integral mean* 0, *then for any finite coefficient system* $\{c_k\}_{k=1}^{n}$ *the following inequality is true*:

$$\int_0^1 \Bigl[\underset{(1 \leq m \leq n)}{\operatorname{Max}_m} \Bigl|\sum_{k=1}^{m} c_k \varphi_k(x)\Bigr|\Bigr]^2 dx \leqq 8 \int_0^1 \Bigl[\sum_{k=1}^{n} c_k \varphi_k(x)\Bigr]^2 dx = 8 \sum_{k=1}^{n} c_k^2. \tag{2.2}$$

In order to prove the convergence of the series (1. 4) it is enough to show that under the conditions (2.1) the series

$$\pi^*(A(x), B(y)) = \sum_{(k, l) \in N^*} a_k b_l r_k(x) r_l(y), \tag{2.3}$$

arising from (1. 4) by omitting brackets [2]), converges almost everywhere on Q, too.

Let us consider for each index n the subseries

$$S_n(x, y) = \sum_{m=1}^{\infty} a_n b_{\nu(n, m)} r_n(x) r_{\nu(n, m)}(y) \tag{2.4}$$

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[1]) A system $\{f_k(x)\}_{k=1}^{n}$ of measurable functions defined on the interval [0, 1] will be called stochastically independent if for an arbitrary given system of intervals $I_k = (\alpha_k, \beta_k)$ $(k = 1, \dots, n)$ the equality

$$m\Bigl(\bigcap_{k=1}^{n} E\{f_k \in I_k\}\Bigr) = \prod_{k=1}^{n} m(E\{f_k \in I_k\})$$

is valid, where $E\{f_k \in I_k\}$ means the set of all $x \in [0, 1]$ for which the inequalities $\alpha_k < f_k(x) < \beta_k$ hold and $m(H)$ denotes the Lebesgue measure of the set H.

A sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ $(x \in [0, 1])$ is stochastically independent, if any finite subsequence of it is stochastically independent in the above sense.

From these definitions follows that any rearrangement $\{f_{n_k}\}_{k=1}^{\infty}$ of a stochastically independent system $\{f_n(x)\}_{n=1}^{\infty}$ remains stochastically independent.

[2]) N^* means the sequence of all elements of $N = \{(k, l)\}$ generated by the decomposition $\bigcup_{m=1}^{\infty} N_m = N$ of the method (π).

which contains — *in unaltered order* — all the terms of (2. 3) having the factor $a_n r_n(x)$.

Since the sequence $\{b_{\nu(n,m)}\}_{m=1}^{\infty}$ is a permutation of the original sequence $\{b_l\}_{l=1}^{\infty}$ (generated by the method (π)), therefore we get from the second condition of (2.1) that

$$\sum_{m=1}^{\infty} a_n^2 b_{\nu(n,m)}^2 = a_n^2 \sum_{m=1}^{\infty} b_{\nu(n,m)}^2 = a_n^2 \sum_{l=1}^{\infty} b_l^2 < \infty$$

is valid for each index n.

This inequality guarantees for each n the existence of such a sequence

$$1 < m_1^{(n)} < m_2^{(n)} < \ldots < m_j^{(n)} < m_{j+1}^{(n)} < \ldots, \tag{2.5}$$

for which the inequalities

$$\sum_{m=m_j^{(n)}}^{\infty} b_{\nu(n,m)}^2 < \frac{1}{4^n} \cdot \frac{1}{4^j} \qquad (j = 1, 2, \ldots)$$

are true, and therefore, using for each n the notation $m_0^{(n)} = 1$ we get from (2.1) the following estimate:

$$\Omega = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \sqrt{\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2} \equiv \Omega_1 + \Omega_2 \equiv$$

$$\equiv \sum_{n=1}^{\infty} \left(\sum_{m=1}^{m_1^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} \leqq \tag{2.6}$$

$$\leqq \sum_{n=1}^{\infty} |a_n| \left(\sum_{m=1}^{m_1^{(n)}-1} b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} |a_n| \frac{1}{2^n} \sum_{j=1}^{\infty} \frac{1}{2^j} \leqq \sqrt{B'} \sum_{n=1}^{\infty} |a_n| + \sqrt{A'}\sqrt{C'} < \infty,$$

where A', B' and C' mean the sums of the convergent series Σa_k^2, Σb_l^2, and $\Sigma\, 4^{-n}$, respectively [3]).

Secondly we construct from the series (2. 4) by the help of the sequences (2. 5) the following series

$$S_n^*(x, y) = \sum_{j=0}^{\infty} \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n b_{\nu(n,m)} r_n(x) r_{\nu(n,m)}(y) \right) \equiv \sum_{j=0}^{\infty} F_j^{(n)}(x, y),$$

and let us denote for each pair (j, n) of indices a general segment of $F_j^{(n)}(x, y)$ by

$$\{F_j^{(n)}(x, y)\}_\nu^\mu = \sum_{m=\nu}^{\mu} a_n b_{\nu(n,m)} r_n(x) r_{\nu(n,m)}(y) \quad (m_j^{(n)} \leqq \nu \leqq \mu \leqq m_{j+1}^{(n)} - 1).$$

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[3]) (2. 6) shows that the condition $\Sigma|a_n| < \infty$ was only used for the estimate $\Omega_1 < \infty$, and that the weaker condition $\Sigma a_n^2 < \infty$ is enough to ensure the validity of $\Omega_2 < \infty$.

Taking into account that for each n and for each $x \in [0, 1]$ the inequalities $|r_n(x)| \leqq 1$ hold, we get for each quartet (j, n, ν, μ) of indices and for each point $(x, y) \in Q$ the following inequality:

$$|\{F_j^{(n)}(x, y\}_\nu^\mu| = |r_n(x)| \left| \sum_{m=\nu}^{\mu} a_n b_{\nu(n,m)} r_{\nu(n,m)}(y) \right| \leqq \tag{2.7}$$
$$\leqq \underset{(m_j^{(n)} \leqq \varrho \leqq m_{j+1}^{(n)}-1)}{\underset{\varrho}{2\mathrm{Max}}} \left| \sum_{m=m_j^{(n)}}^{\varrho} a_n b_{\nu(n,m)} r_{\nu(n,m)}(y) \right| = 2\delta_j^{(n)}(y).$$

Since the Rademacher functions $\{r_{\nu(n,m)}(y)\}_{m=1}^\infty$ evidently satisfy the conditions of Lemma 1, so the functions $\delta_j^{(n)}(y)$ satisfy, according to (2.2), the following integral inequalities:

$$\int_0^1 [\delta_j^{(n)}(y)]^2\, dy \leqq 8 \sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2. \tag{2.8}$$

Introducing the non-negative functions

$$\Delta_j^{(n)}(x, y) = 2\delta_j^{(n)}(y)$$

defined for each pair (j, n) of indices in the unit square Q, we can deduce from (2. 7 and (2. 8) the following two properties:

$$|\{F_j^{(n)}(x, y)\}_\nu^\mu| \leqq \Delta_j^{(n)}(x, y) \text{ for each pair } (j, n) \text{ of indices,} \tag{2.9a}$$

$$\iint_Q [\Delta_j^{(n)}(x, y)]^2\, dx\, dy = 4 \int_0^1 [\delta_j^{(n)}(y)]^2\, dy \leqq 32 \sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2. \tag{2.9b}$$

Using the Schwarz inequality and the rearrangement theorem of series with positive terms [6] we get from (2. 6) and (2. 9b) that

$$\sum_{\lambda=1}^\infty \left(\sum_{j+n=\lambda} \iint_Q \Delta_j^{(n)}(x, y)\, dx\, dy \right) \leqq \sum_{\lambda=1}^\infty \left(\sum_{j+n=\lambda} \left[\iint_Q \{\Delta_j^{(n)}(x, y)\}^2\, dx\, dy \right]^{\frac{1}{2}} \right) =$$
$$= \sum_{n=1}^\infty \sum_{j=0}^\infty \left(\iint_Q [\Delta_j^{(n)}(x, y)]^2\, dx\, dy \right)^{\frac{1}{2}} \leqq \sqrt{32} \sum_{n=1}^\infty \sum_{j=0}^\infty \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right) = \sqrt{32}\,\Omega < \infty,$$

and so, in consequence of the Beppo Levi theorem, the series

$$\sum_{\lambda=1}^\infty \sum_{j+n=\lambda} \Delta_j^{(n)}(x, y) \tag{2.10}$$

converges almost everywhere on Q.

Finally writing consecutive indices in the series (2. 3) we get that each segment

$$\sum_{\tau=p}^{q} \varphi_\tau(x, y) \tag{2.11}$$

of the series

$$\sum_{\tau=1}^{\infty} \varphi_\tau(x, y) \equiv \sum_{(k,l)\in N^*} a_k b_l r_k(x) r_l(y) \equiv \pi^*(A(x), B(y)) \tag{2.12}$$

is a sum of finitely many $\{F_j^{(n)}(x, y)\}_\nu^\mu$, because we preserved the order of the terms of (2. 12) when forming the series (2. 4). Choosing therefore the lower index p in (2. 11) so large that among the terms of the sum

$$\sum_{\tau=1}^{p-1} \varphi_\tau(x, y)$$

every term $a_k b_l r_k(x) r_l(y)$ of the finite sum

$$\sum_{\lambda=1}^{M} \sum_{j+n=\lambda} F_j^{(n)}(x, y)$$

occurs, then we get by (2. 9a) from the convergence of the series (2. 10):

$$\left|\sum_{\tau=p}^{q} \varphi_\tau(x, y)\right| \leqq \sum_{\lambda=M+1}^{\infty} \left(\sum_{j+n=\lambda} \Delta_j^{(n)}(x, y)\right) \to 0$$

when p and $q \to \infty$, which proves the convergence of the series (2. 12), q.e.d.

3.

Let be given two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_l\}_{l=1}^\infty$ satisfying the conditions

$$\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{and} \quad \sum_{l=1}^{\infty} b_l^2 < \infty, \tag{3.1}$$

respectively, and let us consider a given method (π) defined in Definition 1.

Since the product series

$$\pi(\sum a_k^2, \sum b_l^2) = \sum_{m=1}^{\infty} \left(\sum_{(k,l)\in N_m} a_k^2 b_l^2\right) = (\sum a_k^2)(\sum b_l^2)$$

converges [6], there exists an increasing sequence $\{m_\nu\}_{\nu=1}^\infty$ of indices, such that

$$\sum_{\nu=0}^{\infty} \left(\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} a_k^2 b_l^2\right)^{\frac{1}{2}} = \Lambda < \infty \qquad (m_0 = 1) \tag{3.2}$$

holds.

In the light of this fact let us put into brackets the terms of the product series

$$\pi(A(x), B(y)) = \sum_{m=1}^{\infty}{}' \left(\sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \right)$$

of the almost everywhere convergent Rademacher series $\Sigma\, a_k r_k(x)$ and $\Sigma\, b_l r_l(y)$ by the help of the sequence $\{m_\nu\}_{\nu=0}^{\infty}$, i.e. we consider the series

$$\pi^*(A(x), B(y)) = \sum_{\nu=0}^{\infty} \left[\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \right] \equiv \sum_{\nu=0}^{\infty} \Phi_\nu(x, y).$$

We assert that last series converges absolutely almost everywhere on the unit square Q. For our purpose it is enough to observe that the functions

(3. 3) $$R_{k,l}(x, y) = r_k(x) r_l(y) \qquad (k=1, 2, \ldots;\ l=1, 2, \ldots)$$

are orthonormal on Q and so by Schwarz inequality we get from (3. 2) that

$$\sum_{\nu=0}^{\infty} \iint_Q |\Phi_\nu(x, y)|\, dx\, dy \leqq \sum_{\nu=0}^{\infty} \left(\iint_Q \Phi_\nu^2(x, y)\, dx\, dy \right)^{\frac{1}{2}} = \Lambda,$$

which proves our assertion and the following

Theorem 2. *If the conditions* (3. 1) *are fulfilled, then for every method* (π) *we can choose an increasing sequence of indices such that the associated product series*

$$\pi^*(\Sigma \pm a_k, \Sigma \pm b_l) = \sum_{\nu=0}^{\infty} \left[\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} (\pm a_k)(\pm b_l) \right]$$

converges absolutely for almost all signings of the factor series.

Note. If the functions (3. 3) had been stochastically independent on Q, then applying the two-dimensional form of Lemma 1 we should have proved from (3. 2) the perfectness of every method (π) in the strict sense.

In the sequel we indicate a rather interesting problem in the theory of Walsh series, which is essentially equivalent to the problem of the perfectness of general methods (π).

To this end we introduce a convenient form of a famous transformation due to F. Riesz [7], [8].

Before all we co-ordinate the unit interval $I = I_0 = \{t;\ 0 \leqq t \leqq 1\}$ with the unit square $Q = Q_0 = \{(x, y);\ 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$, in sign:

0°) $$I_0 \leftrightarrow Q_0.$$

In the first step we decompose the interval I_0 by the points $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ to four closed intervals $I_{1,1}, I_{1,2}, I_{1,3}, I_{1,4}$, and similarly we divide the square Q_0 by the help of the straight lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ into four congruent closed subsquares $Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}$.

In the case of the intervals $\{I_{1,k}\}_{k=1}^{4}$ the increasing order of the second index k corresponds to the increasing direction of the variable $t \in I_0$ and in the case of the subsquares $\{Q_{1,k}\}_{k=1}^{4}$ the increasing order of the second index is indicated by the scheme in figure 1, i.e. the subsquares $\{Q_{1,k}\}_{k=1}^{4}$ are represented by figure 2, and we order the elements of the systems $\{I_{1,k}\}_{k=1}^{4}$ and $\{Q_{1,k}\}_{k=1}^{4}$ mutually to each other, in sign

1°) $$I_{1,k} \leftrightarrow Q_{1,k} \qquad (k = 1, 2, 3, 4).$$

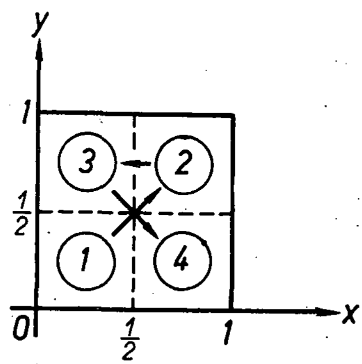

Figure 1

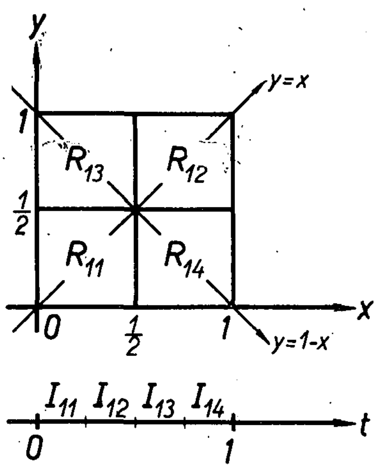

Figure 2

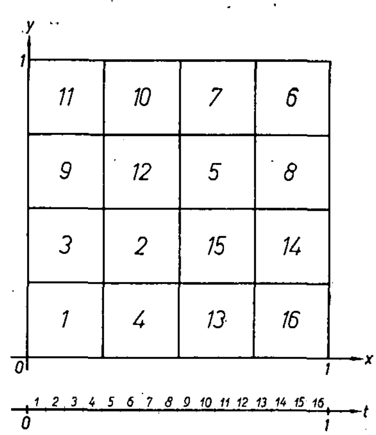

Figure 3

Next we decompose each interval $I_{1,k}$, resp. each quarter $Q_{1,k}$, into four congruent and closed subintervals, resp. subsquares, and we denote the so created systems by

(3.4) $$I_{2,1}, I_{2,2}, \ldots, I_{2,16}; \quad Q_{2,1}, Q_{2,2}, \ldots, Q_{2,16}.$$

In (3. 4) the increasing order of the second indices of the intervals $\{I_{2,k}\}_{k=1}^{16}$ corresponds to the increasing direction of the variable t, and the second indices of the squares $\{Q_{2,k}\}_{k=1}^{16}$ are given so, that the one-to-one mapping

2°) $$I_{2,k} \leftrightarrow Q_{2,k} \qquad (k = 1, 2, ..., 16)$$

has the following two properties:

α) if $I_{2,l} \subset I_{1,k}$ then $Q_{2,l} \subset Q_{1,k}$,

β) if the squares $Q_{2,n}, Q_{2,m}, Q_{2,p}, Q_{2,r}$ are subsquares of a square $Q_{1,k}$, then the increasing order of the second indices is as indicated by the direction scheme in Figure 1.

The mapping 2°) is illustrated by figure 3, where only the second indices are written out in the corresponding subintervals and subsquares.

Iterating the steps 0°), 1°), 2°) periodically we get a sequence $\mathscr{I}$ of intervals

(3.5) $$\mathscr{I} = \{I_0; I_{1,1}, I_{1,2}, I_{1,3}, I_{1,4}; \ldots, I_{n,1}, I_{n,2}, I_{n,4^n}; \ldots\}$$

and a sequence $\mathscr{Q}$ of squares

(3.6) $$\mathscr{Q} = \{Q_0; Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}; \ldots, Q_{n,1}, Q_{n,2}, \ldots, Q_{n,4^n}; \ldots\}$$

for which the one-to-one mapping of the elements

(3.7) $$I_{n,k} \leftrightarrow Q_{n,k} \qquad (n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots, 4^n)$$

has the following three properties:

(I) for each index $n(=1, 2, \ldots)$, $\bigcup_{k=1}^{4^n} I_{n,k} = I_0$ and $\bigcup_{k=1}^{4^n} Q_{n,k} = Q_0$;

(II) for each pair (n, k) of indices there exists such an index l, for which (3.8) $I_{n,k} \subset I_{n-1,l}$ and in this case $Q_{n,k} \subset Q_{n-1,l}$ is valid, and conversely if $Q_{n,k} \subset Q_{n-1,l}$ then $I_{nk} \subset I_{n-1,l}$;

(III) $m(I_{n,k}) = m(Q_{n,k})$, i.e. (3. 7) preserves the measure.

The mapping (3. 7) of the sequences (3. 5) and (3. 6) generates a correspondence between the points of the unit interval I and the unit square Q in the following way:

Definition 6. To each value $t \in I$ let correspond the point (or points) $T(t) = (x, y) \in Q$ for which

(3.9) $$T(t) = (x, y) = \bigcap_{n=1}^{\infty} Q_{n,v(n)}, \quad \text{if} \quad t = \bigcap_{n=1}^{\infty} I_{n,v(n)},$$ [4)]

[4)] If t is not dyadic rational, then the subsequence $\{I_{n,v(n)}\}_{n=1}^{\infty}$ of (3. 5) is uniquely determined by t, and so $T(t) = (x, y) \in Q$ is also uniquely determined.

and conversely to each point $(x, y) \in Q$ let correspond the point (or points) $V(x, y) = t \in I$ for which

(3.10) $$V(x, y) = t = \bigcap_{n=1}^{\infty} I_{n,\mu(n)}, \quad \text{if} \quad (x, y) = \bigcap_{n=1}^{\infty} Q_{n,\mu(n)}.$$

Considering the properties (I)—(III) of the mapping (3. 7) it is easy to see the validity of

Theorem 3. *The transformations* (3. 9) *and* (3. 10) *are inverse of one another apart from sets of measure zero, and both of them are measure preserving transformations.*

Definition 7. The functions $f(t)\,(t \in I)$ and $g(x, y)$ $\big((x, y) \in Q\big)$ will be called equivalent, in sign

(3.11) $$f(t) \simeq g(x, y),$$

if for almost all pairs of corresponding points $\big(t = V(x, y),\ (x, y) = T(t)\big)$ the equality $f(t) = g(x, y)$ holds.

Theorem 3 has the following two corollaries:

Theorem 4. *The function $f(t)$ is measurable resp. integrable on I if and only if the equivalent function $g(x, y)$ is measurable resp. integrable on Q, and in the latter case*

$$\int_0^1 f(t)\, dt = \iint_Q g(x, y)\, dx\, dy.$$

Theorem 5. *If the elements of the sequences $\{g_k(x, y)\}_{k=1}^{\infty}$ $\big((x, y) \in Q\big)$ and $\{f_k(t)\}_{k=1}^{\infty}$ $(t \in I)$ are term by term equivalent in the sense of definition 7, then the series $\sum_{k=1}^{\infty} g_k(x, y)$ converges almost everywhere on Q if and only if the series $\sum_{k=1}^{\infty} f_k(t)$ is convergent almost everywhere on I.*

Finally considering our direction scheme in figure 1, it is easy to see by induction the following.

Theorem 6. *If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of the Rademacher functions, then for the functions of two variables*

$$\varrho_k(x, y) \equiv r_k(x); \quad \sigma_l(x, y) \equiv r_l(y) \qquad (k = 1, 2, \dots\ l = 1, 2, \dots)$$

defined on Q, the following relations are true:

$$\varrho_k(x, y) \simeq r_{2k}(t) \quad \text{and} \quad \sigma_l(x, y) \simeq r_{2l}(t) r_{2l-1}(t) \qquad (k = 1, 2, \dots\ l = 1, 2, \dots)$$

in the sense of Definition 7.

By means of Theorems 5 and 6 we can join the theory of multiplication of infinite series with the theory of Walsh series [9].

In defining the functions of the Walsh system it is convenient to follow PALEY's modification [10]:

Definition 8. If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of Rademacher functions defined in (1. 3) then the Walsh functions $\{w_n(t)\}_{n=0}^{\infty}$ are given in the following form:

$$w_0(t) \equiv 1, \qquad (3.12)$$
$$w_n(t) = r_{\nu_1+1}(t)\, r_{\nu_2+1}(t) \dots r_{\nu_k+1}(t)$$

for $n = 2^{\nu_1} + 2^{\nu_2} + \dots + 2^{\nu_k}$, where the integers ν_i are uniquely determined by $\nu_{i+1} < \nu_i$.

Definition 9. Let $\{w_{n_i}(t)\}_{i=1}^{\infty}$ be such a subsequence of the sequence $\{w_n(t)\}_{n=0}^{\infty}$ of the Walsh functions, whose elements have at most three factors in the Paley representation (3. 12), i.e. whose elements can be written in the form

$$w_{n_i}(t) = r_{\nu_1+1}(t)\, r_{\nu_2+1}(t)\, r_{\nu_3+1}(t).$$

Conjecture. If $\sum c_i^2 < \infty$ then the lacunary Walsh series

$$\sum_{i=1}^{\infty} c_i w_{n_i}(t) \qquad (3.13)$$

converges unconditionally [5]) almost ewerywhere.

Theorem 7. *Let the sequences* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_l\}_{l=1}^{\infty}$ *satisfy the conditions* (3. 1) *and let us consider an arbitrary method* (π) *defined in Definition 1. If the above conjecture is true then the product series*

$$\pi\left(\sum a_k r_k(x), \sum b_l r_l(y)\right) = \sum_{m=1}^{\infty} \left(\sum_{(k,l) \in N_m} a_k b_l r_k(x) r_l(y) \right) \qquad (3.14)$$

converges almost everywhere on the unit square Q.

Proof. From the conditions (3. 1) it follows that the sum

$$\sum_{m=1}^{\infty} \left(\sum_{(k,l) \in N_m} a_k^2 b_l^2 \right)$$

converges, and according to the theorems 5 and 6 the series (3. 14) is equiconvergent with the Walsh series

$$\sum_{m=1}^{\infty} \left(\sum_{(k,l) \in N_m} a_k b_l r_{2k}(t) r_{2l}(t) r_{2l-1}(t) \right),$$

which is a rearranged and associated series of type (3. 13).

[5]) i. e. the series (3.13) converges by any rearrangement of its terms apart from a set of measure zero (which set may depend of course on the rearrangement in question).

Note. The conjecture is true in that special case, when in (3. 13) the indices n_i are in an increasing order; more generally the following theorem is true [11]:

Theorem 8. *If $\{w_{n_i}(x)\}_{i=1}^{\infty}$ denotes such a subsequence of the sequence $\{w_n(x)\}_{n=0}^{\infty}$ whose elements have at most N factors in Paley's representation* (3. 12), *where N is an arbitrary but fixed natural number, then the convergence of the series Σc_i^2 involves the convergence almost everywhere of the lacunary Walsh series*

$$\sum c_i w_{n_i}(x). \tag{3.15}$$

Proof. In order to prove the theorem it is enough to show that the following assertion is true:

If for a given Walsh series $\sum_{n=0}^{\infty} c_n w_n(x)$ the coefficients satisfy the condition $\sum c_n^2 < \infty$ and if the elements of the monotonically increasing sequence n_k have a dyadic expression of the form

$$n_k = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_l}, \tag{3.16}$$

l being limited by an arbitrary chosen but fixed natural number N, then the partial sums

$$s_{n_k}(x) = \sum_{\lambda=0}^{n_k} c_\lambda w_\lambda(x)$$

converge almost everywhere.

We can get the proof of the last assertion from a lemma of L. Leindler [12] applying it to the Walsh system whose Lebesgue constants are known [13].

Lemma 2. *If $\{n_k\}$ is a positive non-decreasing sequence of indices and $\{\varphi_n(x)\}_{n=0}^{\infty}$ is such an orthonormal system on the interval $[a, b]$ whose Lebesgue functions*

$$L_{n_k}(x) = \int_a^b \left| \sum_{i=0}^{n_k} \varphi_i(x) \varphi_i(u) \right| du$$

are uniformly bounded on a set $E \subset [a, b]$, then the condition $\sum c_i^2 < \infty$ implies that the subsequence

$$s_{n_k}(x) = \sum_{i=0}^{n_k} c_i \varphi_i(x)$$

of the partial sums of the orthogonal series $\sum c_i \varphi_i(x)$ is almost everywhere convergent on the set E.

If the orthonormal system $\{\varphi_n(x)\}$ is the Walsh system $\{w_n(x)\}_{n=0}^{\infty}$, then we can apply the above lemma in a very convenient form to the case of the index sequence (3. 16). In fact, N. J. FINE [13] showed that the Lebesgue functions of the Walsh system do not depend on x and have the following explicit form

$$L_n(x) \equiv L_n = l - \sum_{1 \leqq p < r \leqq l} 2^{(\nu_r - \nu_p)}$$

if $n = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_l}$. Consequently, for the indices (3. 16) we have $L_{n_k} \leqq l \leqq N$ and therefore our theorem is proved.

4

Theorem 9. *If the conditions*

$$\sum a_k^2 < \infty \quad \textit{and} \quad \sum b_l^2 < \infty$$

hold, and if the general product series

$$\pi\big(A(x), B(y)\big) = \sum_{m=1}^{\infty} \left(\sum_{(k,l) \in N_m} a_k b_l r_k(x) r_l(y) \right) \tag{4.1}$$

of the almost everywhere convergent series

$$A(x) = \sum_{k=1}^{\infty} a_k r_k(x) \quad \textit{and} \quad B(y) = \sum_{l=1}^{\infty} b_l r_l(y) \tag{4.2}$$

converges almost everywhere on the unit square Q to the sum $S(x, y)$, then $S(x, y) = A(x)B(y)$ almost everywhere on Q.

Proof. Since the rectangular product

$$\sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} a_m b_l r_m(x) r_l(y) + \sum_{k=1}^{m-1} a_k b_m r_k(x) r_m(y) \right)$$

of the series (4. 2) converges almost everywhere on Q to the sum $A(x)B(y)$, and since the product series (4. 1) is, according to (3. 3), such an orthogonal series on Q for which the square sum of its coefficients is finite, so the theorem is an immediate consequence of the Riesz—Fischer theorem.

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(Received September 22, 1967)