On a problem of summability of orthogonal series

By FERENC MÓRICZ and KÁROLY TANDORI in Szeged

Introduction

Let $\{\varphi_k(x)\}$ $(k=0, 1, \dots)$ be an orthonormal system on the finite interval (a, b). We shall denote by $s_n(x)$ the n-th partial sum of the orthogonal series

$$\sum_{k=0}^{\infty} a_k \varphi_k(x). \tag{1}$$

Let $T=(\alpha_{ik})$ $(i, k=0, 1, \dots)$ be a double infinite matrix of numbers. The sum

$$t_i(x) = \sum_{k=0}^{\infty} \alpha_{ik} s_k(x) \qquad (i = 0, 1, \dots)$$

is called the i-th *T-mean* of the series (1), provided that the series on the right-hand side converges. We say that the series (1) is *T-summable* to the sum s at the point $x_0(\in(a, b))$ if $t_i(x_0)$ exists for all i (perhaps except finitely many of them), and $\lim_{i\to\infty} t_i(x_0)=s$. A T-summation process is said to be *permanent* if $\lim_{n\to\infty} s_n=s$ implies $\lim_{i\to\infty} t_i=s$. The necessary and sufficient conditions for the permanence of a summation process are known. (See ALEXITS [1], p. 65.)

For any given orthonormal system $\{\varphi_k(x)\}$ and for any summation matrix T we shall consider the following functions

$$L_i(T; \{\varphi_k\}; x) = \int_a^b \left| \sum_{k=0}^{\infty} \alpha_{ik} \left(\sum_{l=0}^{k} \varphi_l(x) \varphi_l(t) \right) \right| dt = \int_a^b \left| \sum_{l=0}^{\infty} \left(\sum_{k=l}^{\infty} \alpha_{ik} \varphi_k(x) \varphi_k(t) \right) \right| dt,$$

provided they exist. The function $L_i(T; \{\varphi_k\}; x)$ is called the i-th *Lebesgue function* of the orthonormal system $\{\varphi_k(x)\}$ concerning the T-summation process. The order of magnitude of the Lebesgue functions may, in many cases, be decisive for the convergence problems.

In particular, taking

$$\alpha_{ik} = \frac{1}{i+1} \quad (k = 0, 1, \dots, i), \qquad \alpha_{ik} = 0 \quad (k = i+1, i+2, \dots) \quad (i = 0, 1, \dots),$$

we obtain the classical $(C, 1)$-summation process. Now, we have

$$L_i((C,1);\{\varphi_k\};x)=\int_a^b\left|\sum_{k=0}^{i}\left(1-\frac{k}{i+1}\right)\varphi_k(x)\,\varphi_k(t)\right|dt.$$

In this case KACZMARZ [3] has proved the following theorem:

Let $\{\varphi_k(x)\}$ be an arbitrary orthonormal system in (a, b). If $\{\mu_k\}$ is a positive, non-decreasing number sequence for which the relation

$$\sup_{\nu(x)}\int_a^b\frac{L_{\nu(x)}((C,1);\{\varphi_k\};x)}{\mu_{\nu(x)}}\,dx<\infty \tag{2}$$

holds, where the sup is taken over all the measurable functions $\nu(x)$ assuming only integer values, then

$$\sum_{k=0}^{\infty}a_k^2\,\mu_k<\infty$$

implies the $(C, 1)$-summability of the orthogonal series (1) almost everywhere.

It is obvious that the condition (2) is equivalent to the following one:

$$\sup_i\frac{L_i((C,1);\{\varphi_k\};x)}{\mu_i}\in L(a,b).$$

KACZMARZ formulated this theorem under the condition requiring somewhat more than (2), namely

$$L_i((C,1);\{\varphi_k\};x)=O(\mu_i)\qquad(a\leqq x\leqq b),$$

however, the above sharper assertion can also be obtained from his proof.

KACZMARZ [3] has generalized the above theorem also for the $(C, \beta>0)$-summation. (In this case, we have

$$\alpha_{ik}=\frac{A_{i-k}^{(\beta-1)}}{A_i^{(\beta)}}\quad(k=0,1,\ldots,i),\quad\alpha_{ik}=0\,(k=i+1,i+2,\ldots)\quad(i=0,1,\ldots),$$

where $A_i^{(\beta)}=\binom{i+\beta}{i}$.) (See also TANDORI [8].)

SUNOUCHI [7] and LEINDLER [4] have transferred these results to the Riesz summation of orthogonal series. (In this case

$$\alpha_{ik}=\frac{\lambda_{k+1}-\lambda_k}{\lambda_{i+1}}\quad(k=0,1,\ldots,i),\quad\alpha_{ik}=0\quad(k=i+1,i+2,\ldots)\,(i=0,1,\ldots),$$

where $\{\lambda_i\}$ is a positive, strictly increasing sequence of numbers with $\lambda_0=0$ and $\lambda_n \to \infty$.)

To our knowledge, no analogous theorem for other summation processes is yet proved. The following problem can be quite naturally raised: if for any T-summation process the condition

$$\sup_{v(x)} \int_a^b \frac{L_{v(x)}(T;\{\varphi_k\};x)}{\mu_{v(x)}}\,dx < \infty$$

or the stronger one

(3) $$L_i(T;\{\varphi_k\};x) = O(\mu_i) \qquad (a \leqq x \leqq b)$$

is fulfilled, is then the orthogonal series (1) under the condition $\sum_{k=0}^{\infty} a_k^2 \mu_k < \infty$ with the concerning process summable almost everywhere?

EFIMOV [2] has essentially showed that, under the condition (3), $\sum_{k=0}^{\infty} a_k^2 \mu_k < \infty$ with $\mu_k \to \infty$ does not imply the almost everywhere T-summability of the orthogonal series $\sum_{k=0}^{\infty} a_k \varphi_k(x)$ for every permanent T-summation process. In his proof, however, the condition $\mu_k \to \infty$ is very important one.

In this paper we give a construction in which the condition $\mu_k \to \infty$ is not essential. We are going to deal only with the important special case $\mu_k = 1$ ($k = 0, 1, \ldots$). Our theorem reads as follows:

Theorem. *There exist an orthonormal system* $\{\varphi_k(x)\}$ *in* (0, 1), *a coefficient sequence* $\{c_k\}$, *and a permanent T-summation process such that* $\sum_{k=0}^{\infty} c_k^2 < \infty$ *and the relation*

(4) $$\sup_{v(x)} \int_a^b L_{v(x)}(T;\{\varphi_k\};x)\,dx < \infty$$

holds, where the sup is taken over all the measurable functions $v(x)$ *assuming only integer values, but the orthogonal series*

(5) $$\sum_{k=0}^{\infty} c_k \varphi_k(x)$$

is not T-summable almost everywhere in (0, 1).

The proof of our theorem will be accomplished by a direct construction. The T-summation process occurring in the theorem can be chosen as it was found by MENCHOFF [6] and applied to clarify another question.

It is an open question to prove this theorem under the following stronger condition instead of (4):

$$L_i(T; \{\varphi_k\}; x) = O(1) \qquad (0 \leqq x \leqq 1).$$

This problem seems to be difficult.

§ 1. Lemmas

We require two lemmas to prove our theorem. In the following $C_1, C_2, \dots$ will denote positive absolute constants.

Lemma 1. *Let n be a natural number. Then there exist an orthonormal system $\{\omega_l(x)\}$ $(l=0, 1, \dots, 2^{2^n}-1)$ of step-functions in $(0, 1)$, a coefficient sequence $\{b_l\}$ $(l=0, 1, \dots, 2^{2^n}-1)$, and a simple set $E\,(\subseteqq (0, 1))$* [1] *with the following properties: the integral of each function $\omega_l(x)$ extended over $(0, 1)$ vanishes,*

$$\sum_{l=0}^{2^{2^n}-1} b_l^2 \leqq 1, \tag{6}$$

$$L_{2^{2^n}-1}(\{\omega_l\}; x) = \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(x)\,\omega_l(t) \right| dt \leqq 1 \quad (0 \leqq x \leqq 1), \tag{7}$$

$$|E| = \frac{1}{8},\text{[2]} \tag{8}$$

and

$$\max_{0 \leqq s \leqq 2^{2^n}-1} \left| \sum_{l=0}^{s} b_l \omega_l(x) \right| \geqq C_1 \sqrt{n} \quad \text{if} \quad x \in E. \tag{9}$$

Proof. This lemma have been essentially proved in an earlier paper of Tandori [9]. For the sake of completeness, we give its proof in detail here also.

Let $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ be the n-th Rademacher function $(n=0, 1, \dots)$. Let be $w_0(x) \equiv 1$ in $(0, 1)$; if $k \geqq 1$ and $2^{\nu_1} + 2^{\nu_2} + \dots + 2^{\nu_p}$ $(\nu_1 < \nu_2 < \dots < \nu_p)$ is the dyadic representation of k, then let us put $w_k(x) = r_{\nu_1+1}(x) r_{\nu_2+1}(x) \dots r_{\nu_p+1}(x)$. The Walsh functions $w_k(x)$ $(k=0, 1, \dots)$ defined in this manner are step-functions, orthogonal and obviously normed. It is known (see e.g. Alexits [1], p. 188) that for all natural numbers N

$$L_{2^N-1}(\{w_k\}; x) = \int_0^1 \left| \sum_{k=0}^{2^N-1} w_k(x)\, w_k(t) \right| dt \leqq 1. \tag{10}$$

---

[1] A set E will be said simple if it is the union of finitely many, non-overlapping intervals.

[2] $|E|$ denotes the Lebesgue measure of the set E.

Let a be a natural number and let us consider the functions

$$\varphi_a\left(\frac{l}{2^a};x\right)=\frac{1}{2^{a-1}}\prod_{k=2}^{a}\left(1+r_k\left(\frac{l}{2^a}+\frac{l}{2^{a+1}}\right)r_k(x)\right)\qquad (l=0,1,\dots,2^a-1).$$

It is obvious that the functions $\varphi_a(l/2^a;x)$ are linear combinations of the Walsh functions $w_0(x), w_2(x), \dots, w_{2^a-2}(x)$ and that the following equalities are true:

$$\varphi_a\left(\frac{l}{2^a};x\right)=\begin{cases}1 & \text{if } \dfrac{l}{2^a}<x<\dfrac{l+1}{2^a}, \quad \text{or} \\ & \dfrac{1}{2}+\dfrac{l}{2^a}<x<\dfrac{1}{2}+\dfrac{l+1}{2^a}, \text{ or } \dfrac{l}{2^a}-\dfrac{1}{2}<x<\dfrac{l+1}{2^a}-\dfrac{1}{2}, \\ 0 & \text{elsewhere;}\end{cases}$$

and

$$\int_0^1 \varphi_a^2\left(\frac{l}{2^a};x\right)dx=\frac{1}{2^{a-1}}.$$

Now let us consider the following functions:

$$\Phi_1(0;x)=\varphi_2(0;x);$$

$$\Phi_1(1;x)=r_3(x)\varphi_2(0;x),\quad \Phi_2(1;x)=-r_3(x)r_1(x)\varphi_2(0;x);$$

$$\Phi_1(2;x)=r_4(x)\varphi_3(0;x),\quad \Phi_2(2;x)=-r_1(x)\Phi_1(2;x),$$

$$\Phi_3(2;x)=r_5(x)\varphi_3\left(\frac{1}{2^3};x\right),\quad \Phi_4(2;x)=-r_1(x)\Phi_3(2;x);$$

generally,

$$\Phi_{2j+1}(k;x)=r_{2+2^{k-1}+j}(x)\sum_l \varphi_{2+2^{k-2}+[j/2]}(x_l;x)\quad (j=0,1,\dots,2^{k-1}-1),\ ^{3)}$$

where the points x_l denote the left-hand side endpoints of the subintervals of $(0,\frac{1}{2})$, in which the function $\Phi_{j+1}(k-1;x)$ is positive, and finally

$$\Phi_{2j}(k;x)=-r_1(x)\Phi_{2j-1}(k;x)\qquad (j=1,2,\dots,2^{k-1}).$$

It is clear that for an arbitrary natural number $n(\geqq 2)$ the functions $\Phi_r(k;x)$ $(k=0,1,\dots,n-1;\ r=1,2,\dots,2^k)$ possess the following properties: these functions are also linear combinations of the Walsh functions, namely

$$\Phi_r(k;x)=\sum_i b_i(r,k)\,w_{n(i,r,k)}(x)\qquad (n(1,r,k)<n(2,r,k)<\dots); \tag{11}$$

3) $[\alpha]$ denotes the integer part of α.

the different functions $\Phi_r(k; x)$ have no common Walsh function in their representation (11); in this representation of the function $\Phi_r(k; x)$ $(k=0, 1, \ldots, n-1;$ $r=1, 2, \ldots, 2^k)$ only some of the Walsh functions $w_0(x), w_1(x), \ldots, w_{2^{2^{n-1}}+2^{2^{n-2}+1}-1}(x)$ occur; furthermore, the inequality

$$(12) \qquad \sum_{r=1}^{2^k} \int_0^1 \Phi_r^2(k; x)\, dx \leqq 1 \qquad (k=0, 1, \ldots, n-1)$$

is satisfied.

Now, let us consider the following sum:

$$S_n(x) = \Phi_1(0; x) + \sum_{k=1}^{n-1} \sum_{j=0}^{2^{k-1}-1} \left(\Phi_{2j+1}(k; x) + 2\Phi_{2(j+1)}(k; x)\right) =$$

$$= \sum_{l=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_l(n)\, w_l(x).$$

On account of (12) we get

$$(13) \qquad \int_0^1 S_n^2(x)\, dx = \sum_{l=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_l^2(n) \leqq 5n.$$

Finally, set us arrange the terms $\Phi_j(k; x)$ of the sum $S_n(x)$ by recurrence with respect to k: let

$$s_1(S_n; x) = \Phi_1(0; x) + \Phi_1(1; x) + 2\Phi_2(1; x),$$

$$s_2(S_n; x) = \Phi_1(0; x) + \Phi_1(1; x) + \Phi_1(2; x) + 2\Phi_2(2; x) +$$

$$+ 2\Phi_2(1; x) + \Phi_3(2; x) + 2\Phi_4(2; x),$$

and so on. In general, from $s_\mu(S_n; x)$ we obtain $s_{\mu+1}(S_n; x)$ in such a manner that for every term $\Phi_{2j+1}(\mu; x)$ and $\Phi_{2(j+1)}(\mu; x)$ $(j=0, 1, \ldots, 2^{\mu-1}-1)$ we look for the place where they occur in $s_\mu(S_n; x)$, and then immediately after them we insert the sums $\Phi_{2^2 j+1}(\mu+1; x) + 2\Phi_{2^2 j+2}(\mu+1; x)$ and $\Phi_{2^2 j+3}(\mu+1; x) + 2\Phi_{2^2 j+4}(\mu+1; x)$, respectively. Now, let us choose the set $\bar{E}$ that is the set of the points of the interval $(0, \frac{1}{4})$ at which $w_l(x) \neq 0$ $(l=0, 1, \ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1)$ (i.e. apart from a finite number of the dyadically rational points). It is clear that this $\bar{E}$ is a simple set and $|\bar{E}| = \frac{1}{4}$. From the definition of $\Phi_r(k; x)$ we get that the maximum of the partial sums of the prescribed rearrangement of the sum $S_n(x)$ will equal n in the points of $\bar{E}$.

If we substitute the representations (11) in the above rearrangement of $S_n(x)$

and label the occurring Walsh functions, in this order, by the subscript n_i ($i=0, 1, \ldots$ $\ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1$) then we have

$$S_n(x) = \sum_{i=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_{n_i}(n)\, w_{n_i}(x).$$

Then the above assertion may be written as follows:

$$(14)\qquad \max_{1\leqq s\leqq 2^{2^{n-1}}+2^{2^{n-2}+1}-1} \left|\sum_{i=0}^{s} b_{n_i}(n)\, w_{n_i}(x)\right| = n \qquad (x\in \bar{E}).$$

Now we put

$$\bar{\omega}_i(x) = w_{n_i}(x) \qquad (i = 0, 1, \ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1),$$
$$\bar{\omega}_i(x) = w_i(x) \qquad (i = 2^{2^{n-1}}+2^{2^{n-2}+1}, \ldots, 2^{2^n}-1);$$
$$b_i = \frac{b_{n_i}(n)}{\sqrt{5n}} \qquad (i = 0, 1, \ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1),$$
$$b_i = 0 \qquad (i = 2^{2^{n-1}}+2^{2^{n-2}+1}, \ldots, 2^{2^n}-1).$$

This is possible as $2^{2^{n-1}}+2^{2^{n-2}+1}-1 \leqq 2^{2^n}-1$. Finally, we set

$$\omega_i(x) = \begin{cases} \bar{\omega}_i(2x) & \text{if } x\in \left(0, \frac{1}{2}\right), \\ -\bar{\omega}_i(2x-1) & \text{if } x\in \left(\frac{1}{2}, 1\right), \\ 0 & \text{elsewhere}, \end{cases}$$

($i=0, 1, \ldots, 2^{2^n}-1$). Furthermore, let E be the set arising from $\bar{E}$ as the result of the linear transformation of the interval (0, 1) into the subinterval (0, $\frac{1}{2}$).

It is obvious that E is a simple set and the assertion (8) is satisfied. We can easily see that the function system $\{\bar{\omega}_i(x)\}$ ($i=0, 1, \ldots, 2^{2^n}-1$) is a rearrangement of the Walsh functions $\{w_i(x)\}$ ($i=0, 1, \ldots, 2^{2^n}-1$). From (10) we have

$$\int_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \bar{\omega}_l(x)\, \bar{\omega}_l(t)\right| dt \leqq 1 \qquad (0 \leqq x \leqq 1).$$

Hence, by a simple calculation we get that assertion (7) is also satisfied. Furthermore, by virtue of (13) and (14), the inequalities (6) and (9) hold. Finally, taking into account the construction of $\omega_l(x)$ it is obvious that

$$\int_0^1 \omega_l(x)\, dx = 0 \qquad (l = 0, 1, \ldots, 2^{2^n}-1).$$

The proof is thus completed.

Lemma 2. *Let n be a natural number, λ real number such that $0<\lambda<1$, furthermore, let I_1, I_2, I_3 be arbitrary, mutually disjoint subintervals of the interval $(0, 1)$ for which $|I_2| \leqq |I_1|$ and $|I_3| \leqq |I_1|$ are satisfied. Then there exist an orthonormal system $\{\psi_k(x)\}$ $(k=1, 2, \ldots, 2.2^{2^n})$ of step-functions in $(0, 1)$, a coefficient sequence $\{d_k\}$ $(k=1, 2, \ldots, 2.2^{2^n})$, and a simple set $F(\subseteqq I_1)$ having the following properties: the integral of each function $\psi_k(x)$ extended over $(0, 1)$ vanishes,*

$$\sum_{k=1}^{2.2^{2^n}} d_k^2 \leqq 1 \qquad (d_k = 0 \quad \text{if} \quad k = 2^{2^n}+1, \ldots, 2.2^{2^n}), \tag{15}$$

$$|F| = \frac{|I_1|}{8}, \tag{16}$$

$$\max_{1 \leqq s < 2^{2^n}} \left| \sum_{k=1}^{s} d_k \psi_k(x) \right| \geqq C_2 \lambda \sqrt{n} \quad \text{if} \quad x \in F; \tag{17}$$

for the Lebesgue functions of this system the following upper estimates hold:

$$L_{2^{2^n}}(\{\psi_k\}; x) = \int_0^1 \left| \sum_{k=1}^{2^{2^n}} \psi_k(x) \psi_k(t) \right| dt \leqq \begin{cases} C_3 \lambda & (x \in I_1), \\ C_4/\sqrt{|I_2|} & (x \in I_2), \\ C_5/\sqrt{|I_3|} & (x \in I_3), \\ 0 & \text{elsewhere}; \end{cases} \tag{18}$$

$$L_{2.2^{2^n}}(\{\psi_k\}; x) \leqq \begin{cases} C_6 \lambda & (x \in I_1), \\ C_7/\sqrt{|I_2|} & (x \in I_2), \\ 1 & (x \in I_3), \\ 0 & \text{elsewhere}; \end{cases} \tag{19}$$

furthermore, for the function

$$R_i(x) = \int_0^1 \left| \sum_{k=1}^{i} \psi_k(x) \psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x) \psi_k(t) \right| dt \qquad (1 \leqq i < 2^{2^n})$$

we have also the following upper estimate:

$$R_i(x) \leqq \begin{cases} C_8 \lambda + L_n \sqrt{|I_3|}/\sqrt{|I_1|} & (x \in I_1), \\ C_9 L_n/\sqrt{|I_2|} & (x \in I_2), \\ C_{10} L_n/\sqrt{|I_3|} & (x \in I_3), \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

with

$$L_n = \max_{\substack{0 \leqq i \leqq 2^{2^n}-1, \\ 0 \leqq x \leqq 1}} L_i(\{\omega_l\}; x);$$

the functions $\omega_l(x)$ occurring here are defined by Lemma 1. *(As the functions $\omega_l(x)$ are uniformly bounded, L_n is a finite number for every n.)*

Proof. Let $f(x)$ be an arbitrary function defined in the interval (0, 1), furthermore, let $I=(a, b)$ be an arbitrary subinterval of (0, 1) and H an arbitrary subset of (0, 1). Now, we proceed from the interval (0, 1) to the interval I by means of the linear tarnsformation $y=(x-a)/(b-a)$ $(a \leqq x \leqq b,\ 0 \leqq y \leqq 1)$, and put

$$f(I; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (a \leqq x \leqq b), \\ 0 & \text{elsewhere;} \end{cases}$$

let $H(I)$ be the set into which H is carried over by this linear transformation.

Let $\{\omega_l(x)\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$, $\{b_l\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$ and E denote the corresponding orthonormal system, the coefficient sequence, and the simple set occurring in Lemma 1, respectively.

Let us put

$$d_l = \begin{cases} b_{l-1} & \text{for} \quad 1 \leqq l \leqq 2^{2^n}, \\ 0 & \text{for} \quad 2^{2^n}+1 \leqq l \leqq 2.2^{2^n}; \end{cases}$$

furthermore, $F=E(I_1)$. It then follows from (6) and (8) that (15) and (16) are fulfilled. The functions $\psi_k(x)$ are defined as follows: for $k=1, 2, \ldots, 2^{2^n}$ let us set

$$\psi_k(x) = \frac{\lambda}{\sqrt{2|I_1|}}\omega_{k-1}(I_1; x) + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\omega_{k-1}(I_2; x) + \frac{1}{\sqrt{2|I_3|}}\omega_{k-1}(I_3; x),$$

and for $k = 2^{2^n}+1, \ldots, 2.2^{2^n}$

$$\psi_k(x) = \frac{\lambda}{\sqrt{2|I_1|}}\omega_{2.2^{2^n}-k}(I_1; x) + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\omega_{2.2^{2^n}-k}(I_2; x) - \frac{1}{\sqrt{2|I_3|}}\omega_{2.2^{2^n}-k}(I_3; x).$$

By a simple calculation we get from these definitions that the functions $\psi_k(x)$ form an orthonormal system in (0, 1). If $x \in F$ then there exists $y \in E$ such that

$$\psi_k(x) = \frac{\lambda}{\sqrt{2|I_1|}}\omega_{k-1}(y) \qquad (k = 1, 2, \ldots, 2^{2^n}),$$

thus the correctness of (17) follows from (9). On account of Lemma 1, it is clear that

$$\int_0^1 \psi_k(x)\, dx = 0 \qquad (k = 1, 2, \ldots, 2.2^{2^n}).$$

It remains to be proved that the inequalities (18), (19) and (20) are also satisfied.

First of all, we remark that the functions $\psi_k(x)$ vanish outside the set $I_1 \cup I_2 \cup I_3$. According to the definition of the functions $\psi_k(x)$, by calculating the integrals on the right-hand side, we obtain for $x \in I_1$

$$
\begin{aligned}
L_{2^{2^n}}(\{\psi_k\}; x) &= \frac{\lambda}{\sqrt{2|I_1|}}\left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)\left|\sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_1; x)\psi_k(t)\right| dt = \\
&= \frac{\lambda}{\sqrt{2|I_1|}}\left(\frac{\lambda}{\sqrt{2|I|_1}}|I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2| + \frac{|I_3|}{\sqrt{2|I_3|}}\right)\int_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y)\omega_l(t)\right| dt;^{4)}
\end{aligned}
\tag{21}
$$

for $x \in I_2$

$$
\begin{aligned}
L_{2^{2^n}}(\{\psi_k\}; x) &= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)\left|\sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_2; x)\psi_k(t)\right| dt = \\
&= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\left(\frac{\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2| + \frac{|I_3|}{\sqrt{2|I_3|}}\right)\int_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y')\omega_l(t)\right| dt;
\end{aligned}
\tag{22}
$$

and for $x \in I_3$

$$
\begin{aligned}
L_{2^{2^n}}(\{\psi_k\}; x) &= \frac{1}{\sqrt{2|I_3|}}\left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)\left|\sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_3; x)\psi_k(t)\right| dt = \\
&= \frac{1}{\sqrt{2|I_3|}}\left(\frac{\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2| + \frac{|I_3|}{\sqrt{2|I_3|}}\right)\int_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y'')\omega_l(t)\right| dt.
\end{aligned}
\tag{23}
$$

By paying attention to (7), from (21), (22) and (23) we obtain the estimate (18)

Now we treat the Lebesgue function $L_{2 \cdot 2^{2^n}}(\{\psi_k\}; x)$. We also distinguish three subcases as above. If $x \in I_1$, we get

$$
\begin{aligned}
L_{2.2^{2^n}}(\{\psi_k\}; x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)\left|\sum_{k=1}^{2.2^{2^n}} \psi_k(x)\psi_k(t)\right| dt = \\
&= \frac{\lambda}{\sqrt{2|I_1|}}\left(\frac{2\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{2\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2|\right)\int_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y)\omega_l(t)\right| dt;
\end{aligned}
\tag{24}
$$

---

[4]) Let y, y' and y'' denote the image points into which the points $x \in I_1$, $x \in I_2$ and $x \in I_3$ are carried over by the corresponding linear transformations transferring the intervals I_1, I_2 and I_3 into the interval (0, 1), respectively.

if $x \in I_2$ then

$$L_{2.2^{2^n}}(\{\psi_k\}; x) = \left(\int\limits_{I_1} + \int\limits_{I_2} + \int\limits_{I_3}\right) \left|\sum_{k=1}^{2.2^{2^n}} \psi_k(x)\psi_k(t)\right| dt =$$

(25)

$$= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left(\frac{2\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{2\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2|\right) \int\limits_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y')\omega_l(t)\right| dt;$$

and if $x \in I_3$ then

$$L_{2.2^{2^n}}(\{\psi_k\}; x) = \left(\int\limits_{I_1} + \int\limits_{I_2} + \int\limits_{I_3}\right) \left|\sum_{k=1}^{2.2^{2^n}} \psi_k(x)\psi_k(t)\right| dt =$$

(26)

$$= \frac{1}{2|I_3|} 2|I_3| \int\limits_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y'')\omega_l(t)\right| dt.$$

By virtue of (7), (24), (25) and (26) we have also the estimate (19).

The validity of (20) follows in a similar way as before. According to the definition of the function $R_i(x)$, we have for $x \in I_1$

$$R_i(x) = \left(\int\limits_{I_1} + \int\limits_{I_2} + \int\limits_{I_3}\right) \left|\sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t)\right| dt =$$

(27)

$$= \frac{\lambda}{\sqrt{2|I_1|}} \left\{\left(\frac{\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2|\right) \int\limits_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y)\omega_l(t)\right| dt + \right.$$

$$\left. + \frac{1}{\sqrt{2|I_3|}}|I_3| \int\limits_0^1 \left|\sum_{l=0}^{i-1} \omega_l(y)\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y)\omega_l(t)\right| dt\right\};$$

for $x \in I_2$

$$R_i(x) = \left(\int\limits_{I_1} + \int\limits_{I_2} + \int\limits_{I_3}\right) \left|\sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t)\right| dt =$$

(28)

$$= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left\{\left(\frac{\lambda}{\sqrt{2|I_1|}}|I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}|I_2|\right) \int\limits_0^1 \left|\sum_{l=0}^{2^{2^n}-1} \omega_l(y')\omega_l(t)\right| dt + \right.$$

$$\left. + \frac{1}{\sqrt{2|I_3|}}|I_3| \int\limits_0^1 \left|\sum_{l=0}^{i-1} \omega_l(y')\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y')\omega_l(t)\right| dt\right\};$$

and finally for $x \in I_3$

$$R_i(x) = \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \left| \sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t) \right| dt =$$

$$(29) \quad \frac{1}{\sqrt{2|I_3|}} \left\{ \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{i-1} \omega_l(y'')\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y'')\omega_l(t) \right| dt + \right.$$

$$\left. + \frac{|I_3|}{\sqrt{2|I_3|}} \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y'')\omega_l(t) \right| dt \right\}.$$

Taking into consideration that $|I_2| < 1$, $|I_3| < 1$ and $L_n \geqq 1$, from (27), (28) and (29) we obtain the estimate (20). This completes the proof of Lemma 2.

§ 2. Proof of the theorem

Let $\{\nu_n\}$ and $\{N_n\}$ ($n = 2, 3, \ldots$) be the following sequences of natural numbers:

$$\nu_n = 2^{2^{n^6}} \qquad (n = 2, 3, \ldots),$$

$$N_2 = 0, \quad N_n = \sum_{i=2}^{n-1} 2\nu_i \qquad (n = 3, 4, \ldots).$$

Define the matrix $T = \{\alpha_{ik}\}$ ($i, k = 0, 1, 2, \ldots$) occurring in our theorem as follows:

$$\alpha_{00} = 1, \quad \alpha_{0k} = 0 \qquad (k = 1, 2, \ldots),$$

and in general, for an arbitrary natural number $n (\geqq 2)$ we distinguish three subcases: if $N_n < i < N_n + \nu_n$ then we put

$$\alpha_{ii} = \tfrac{1}{2}, \quad \alpha_{i,\, N_{n+1}-(i-N_n)} = \tfrac{1}{2}, \quad \alpha_{ik} = 0 \qquad \text{otherwise};$$

if $i = N_n + \nu_n$ then

$$\alpha_{i,\, N_n+\nu_n} = 1, \quad \alpha_{ik} = 0 \qquad \text{otherwise};$$

and finally if $N_n + \nu_n < i \leqq N_{n+1}$ then

$$\alpha_{i,\, N_{n+1}} = 1, \quad \alpha_{ik} = 0 \qquad \text{otherwise}.$$

From the definition of the matrix T it immediately follows that the conditions

$$\alpha_{ik} \geqq 0 \quad (i, k = 0, 1, 2, \ldots); \quad \lim_{i \to \infty} \alpha_{ik} = 0 \quad (k = 0, 1, 2, \ldots);$$

$$\sum_{k=0}^{\infty} \alpha_{ik} = 1 \quad (i = 0, 1, 2, \ldots)$$

are satisfied. Therefore, on account of a theorem (see e.g. Alexits [1], p. 65) we infer the permanence of the T-summation process.

To define the orthonormal system $\{\varphi_k(x)\}$ $(k=0, 1, 2, \ldots)$ and the coefficient sequence $\{c_k\}$ $(k=0, 1, 2, \ldots)$ occurring in our theorem we apply induction. The construction is similar to that of TANDORI [10].

Let $\lambda_n = 1/n$ $(n=2, 3, \ldots)$ be. First of all, let us consider three sequences of subintervals $\{I_1(n)\}$, $\{I_2(n)\}$ and $\{I_3(n)\}$ of the interval $(0, 1)$ so that the conditions

$$I_i(n) \cap I_j(n) = O \qquad (i \neq j; \quad n = 2, 3, \ldots); \tag{30}$$

$$\begin{aligned} &I_i(n') \cap I_i(n'') = O \qquad (i = 2, 3; \quad n' \neq n''; \quad n', n'' = 2, 3, \ldots); \\ &I_2(n') \cap I_3(n'') = O \qquad (n', n'' = 2, 3, \ldots); \end{aligned} \tag{31}$$

$$I_1(n) = \left[\frac{2^{m+1}-n}{2^m}, \frac{2^{m+1}-n+1}{2^m}\right) \qquad (2^m < n \leqq 2^{m+1}; \quad m = 0, 1, 2, \ldots); \tag{32}$$

$$\sum_{n=2}^{\infty} L_{n^6}\left(\sqrt{|I_2(n)|} + \sqrt{|I_3(n)|}\right) < \infty, \tag{33}$$

where L_n is defined in Lemma 2, and

$$\frac{L_{n^6}\sqrt{|I_3(n)|}}{\sqrt{|I_1(n)|}} \leqq \lambda_n \qquad (n = 2, 3, \ldots) \tag{34}$$

should be satisfied. It is obvious that both intervals $I_2(n)$ and $I_3(n)$ can be chosen in accordance with these requirements.

From (31) we can easily see that every point x of $(0, 1)$ belongs to at most one of all the subintervals $I_2(n)$ and $I_3(n)$. Furthermore, by (32) it follows that every point $x \in (0, 1)$ lies in $I_1(n)$ for infinitely many values of n, and for every non-negative integer m there exists a uniquely determined natural number $n_m(x)$ for which $2^m < n_m(x) \leqq 2^{m+1}$ and $x \in I_1(n_m(x))$. By the definition of $\{\lambda_n\}$ we get immediately that

$$\sum_{m=0}^{\infty} \lambda_{n_m(x)} \leqq \sum_{m=0}^{\infty} \frac{1}{2^m} = 2. \tag{35}$$

Now we are going to construct a system $\{\varphi_k(x)\}$ $(k=0, 1, 2, \ldots)$ of orthonormal step-functions in $(0, 1)$, a coefficient sequence $\{c_k\}$ $(k=0, 1, 2, \ldots)$, and a sequence of simple subsets $G_n(\subseteq I_1(n))$ $(n=2, 3, \ldots)$ in $(0, 1)$ so that the following relations should be satisfied:

$$\sum_{k=N_n+1}^{N_n+\nu_n} c_k^2 \leqq \frac{1}{n^2} \quad \text{and} \quad c_k = 0 \quad \text{for} \quad k = N_n+\nu_n+1, \ldots, N_{n+1} \qquad (n = 2, 3, \ldots); \tag{36}$$

$$|G_n| = \frac{|I_1(n)|}{8}; \tag{37}$$

$$\max_{N_n < i \leqq N_n+\nu_n} \left| \sum_{k=N_n+1}^{i} c_k \varphi_k(x) \right| \geqq C_2 n \quad \text{if} \quad x \in G_n \qquad (n = 2, 3, \ldots); \tag{38}$$

furthermore,

$$(39)\qquad \int_0^1 \left|\sum_{k=N_n+1}^{N_n+\nu_n} \varphi_k(x)\varphi_k(t)\right| dt \leqq \begin{cases} C_3\lambda_n & (x\in I_1(n)),\\ C_4/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_5/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere}; \end{cases} \quad (n=2,3,\ldots);$$

$$(40)\qquad \int_0^1 \left|\sum_{k=N_n+1}^{N_{n+1}} \varphi_k(x)\varphi_k(t)\right| dt \leqq \begin{cases} C_6\lambda_n & (x\in I_1(n)),\\ C_7/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ 1 & (x\in I_3(n)),\\ 0 & \text{elsewhere}; \end{cases} \quad (n=2,3,\ldots);$$

$$(41)\qquad S_i(n;x) = \int_0^1 \left|\sum_{k=N_n+1}^{N_n+i} \varphi_k(x)\varphi_k(t) + \sum_{k=N_n+\nu_n+1}^{N_{n+1}-i} \varphi_k(x)\varphi_k(t)\right| dt \leqq$$

$$\leqq \begin{cases} C_8\lambda_n + L_{n^6}\sqrt{|I_3(n)|}/\sqrt{|I_1(n)|} & (x\in I_1(n)),\\ C_9 L_{n^6}/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_{10} L_{n^6}/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere} \end{cases}$$

$$(N_n < i < N_n+\nu_n;\quad n=2,3,\ldots).$$

We notice that, on account of (34) and (41), the estimate

$$(42)\qquad S_i(n;x) \leqq \begin{cases} C_{11}\lambda_n & (x\in I_1(n)),\\ C_9 L_{n^6}/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_{10} L_{n^6}/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere} \end{cases}$$

$$(N_n < i < N_n+\nu_n;\quad n=2,3,\ldots)$$

also follows.

Let $\varphi_0(x)\equiv 1$ and $c_0=0$ be. We apply Lemma 2 with $n=2^6$, $\lambda=\lambda_2$ and $I_i=I_i(2)$ $(i=1,2,3)$ (on account of (30) it is permissible). We get the orthonormal system $\{\psi_k(x)\}$ $(k=1,2,\ldots,2\nu_2)$, the coefficient sequence $\{d_k\}$ $(k=1,2,\ldots,2\nu_2)$, and the simple set F satisfying (15)—(20). Now we write

$$\varphi_k(x) = \psi_k(x),\quad c_k = \frac{d_k}{2} \quad (k=1,2,\ldots,N_3),\ \text{and}\quad G_2 = F.$$

According to Lemma 2 the step-functions $\varphi_k(x)$ $(k=0,1,\ldots,N_3)$ are orthonormal, and the relations (36)—(41) hold for $n=2$.

Now, $n_0(\geqq 2)$ being arbitrary, we assume that the step-functions $\varphi_k(x)$ $(k=0,1,\ldots,N_{n_0+1})$, the coefficients c_k $(k=0,1,\ldots,N_{n_0+1})$, and the simple sets $G_n\,(\subseteqq I_1(n))$ $(n=2,3,\ldots,n_0)$ are already determined such that these functions are orthogonal and normed in (0, 1) and that the requirements (36)—(41) are satisfied for each integer $n \leqq n_0$. We are going to construct the functions, coefficients, and simple set corresponding to n_0+1 so that these also satisfy (36)—(41).

We can divide the intervals $I_1(n_0+1)$, $I_2(n_0+1)$ and $I_3(n_0+1)$ into a finite number of mutually disjoint subintervals

$$I_1(n_0+1) = \bigcup_{i=1}^{q_1} J_i(1), \quad I_2(n_0+1) = \bigcup_{i=1}^{q_2} J_i(2), \quad I_3(n_0+1) = \bigcup_{i=1}^{q_3} J_i(3)$$

on which every function $\varphi_k(x)$ $(k=0, 1, \ldots, N_{n_0+1})$ remains constant, and every set $G_n \cap I_1(n_0+1)$ $(n=2, 3, \ldots, n_0)$ can be represented as the union of some intervals $J_i(1)$.

We begin with applying Lemma 1 with $n=(n_0+1)^6$. We get the functions $\omega_l(x)$ $(l=0, 1, \ldots, 2^{2^{(n_0+1)^6}}-1)$. Next applying Lemma 2 with $n=(n_0+1)^6$, $\lambda=\lambda_{n_0+1}$ and $I_i=I_i(n_0+1)$ $(i=1, 2, 3)$, we obtain the functions $\psi_k(x)$ $(k=1, 2, \ldots, 2\nu_{n_0+1})$, the coefficients d_k $(k=1, 2, \ldots, 2\nu_{n_0+1})$, and the simple set F_{n_0+1}. Let us put

$$\varphi_{N_{n_0+1}+l}(x) = \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \omega_{l-1}(J_i(1); x) + \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \omega_{l-1}(J_i(2); x) +$$

$$+ \frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \omega_{l-1}(J_i(3); x) \qquad (l = 1, 2, \ldots, \nu_{n_0+1}),$$

$$\varphi_{N_{n_0+1}+\nu_{n_0+1}+l}(x) = \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \omega_{\nu_{n_0+1}-l}(J_i(1); x) +$$

$$+ \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \omega_{\nu_{n_0+1}-l}(J_i(2); x) - \frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \omega_{\nu_{n_0+1}-l}(J_i(3); x)$$

$$(l = 1, 2, \ldots, \nu_{n_0+1}).$$

It is clear that the functions $\varphi_k(x)$ $(k=N_{n_0+1}+1, \ldots, N_{n_0+2})$ are also step-functions. By virtue of Lemma 1 and the definition, we can easily prove that the functions $\varphi_k(x)$ $(k=0, 1, \ldots, N_{n_0+2})$ are orthonormal in (0, 1).

Let us put

$$c_{N_{n_0+1}+k} = \frac{d_k}{n_0+1} \qquad (k = 1, 2, \ldots, 2\nu_{n_0+1}).$$

From (16) it follows that (36) is satisfied for $n=n_0+1$. Finally, we set

$$G_{n_0+1} = \bigcup_{i=1}^{q_1} E(J_i(1)).$$

It is obvious that G_{n_0+1} is a simple set, and on account of Lemma 1, (37) holds for $n=n_0+1$.

If $x \in G_{n_0+1}$ then there exists a point $y \in F_{n_0+1}$ such that

$$\varphi_{N_{n_0+1}+k}(x) = \psi_k(y) \qquad (k = 1, 2, \ldots, 2\nu_{n_0+1}).$$

Taking into consideration of the definition of the coefficients c_k and (17), we obtain (38) for $n = n_0 + 1$.

According to the definition of the functions $\varphi_k(x)$ $(N_{n_0+1} < k \leqq N_{n_0+2})$ and the proof of Lemma 2, if $x \in (0, 1)$ then for an appropriately chosen y we have

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+1}+\nu_{n_0+1}} \varphi_k(x)\varphi_k(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt.$$

To show this, let $x \in I_1(n_0+1) \cup I_2(n_0+1) \cup I_3(n_0+1)$ be fixed. Then by simple integral transformations we get that the left-hand side equals

$$\frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \int_{J_i(1)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(1); t) \right| dt +$$

$$+ \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \int_{J_i(2)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(2); t) \right| dt +$$

$$+ \frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \int_{J_i(3)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(3); t) \right| dt =$$

$$= \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_1} |J_i(1)| +$$

$$+ \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_2} |J_i(2)| +$$

$$+ \frac{1}{\sqrt{2|I_3(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_3} |J_i(3)| =$$

$$= \left(\int_{J_1(n_0+1)} + \int_{J_2(n_0+1)} + \int_{J_3(n_0+1)} \right) \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt.$$

Here we took into consideration that

$$\sum_{i=1}^{q_1} |J_i(1)| = |I_1(n_0+1)|, \quad \sum_{i=1}^{q_2} |J_i(2)| = |I_2(n_0+1)|, \quad \sum_{i=1}^{q_3} |J_i(3)| = |I_3(n_0+1)|.$$

Similarly, we have also the following equations:

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+2}} \varphi_k(x)\varphi_k(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{2\nu_{n_0+1}} \psi_l(x)\psi_l(t) \right| dt,$$

and

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+1}+i} \varphi_k(x)\varphi_k(t) + \sum_{k=N_{n_0+1}+\nu_{n_0+1}+1}^{N_{n_0+2}-i} \varphi_k(x)\varphi_k(t) \right| dt =$$

$$= \int_0^1 \left| \sum_{l=1}^{i} \psi_l(y)\psi_l(t) + \sum_{l=\nu_{n_0+1}+1}^{2\nu_{n_0+1}-i} \psi_l(y)\psi_l(t) \right| dt \qquad (i = 1, 2, \ldots, \nu_{n_0+1}-1);$$

here $y \in I_1(n_0+1)$, $y \in I_2(n_0+1)$, $y \in I_3(n_0+1)$ and $y \notin \bigcup_{i=1}^{3} I_i(n_0+1)$ according to $x \in I_1(n_0+1)$, $x \in I_2(n_0+1)$, $x \in I_3(n_0+1)$ and $x \notin \bigcup_{i=1}^{3} I_i(n_0+1)$, respectively. By (18), (19) and (20) we get (39), (40) and (41) also for $n = n_0+1$.

Thus we obtained the orthonormal system $\{\varphi_k(x)\}$, the coefficient sequence $\{c_k\}$, and the sequence of simple sets $\{G_n\}$ by induction, which fulfil the requirements (36)—(41).

Let us consider the sets

$$H_m = \bigcup_{n=2^m+1}^{2^{m+1}} G_n \qquad (m = 1, 2, \ldots).$$

By virtue of the definition of the intervals $I_1(n)$ and (36), we have

(43) $$|H_m| = \frac{1}{8} \qquad (m = 1, 2, \ldots).$$

According to the definition of the sets G_n, it can easily be seen that the sets H_m are stochastically independent. Applying the Borel—Cantelli lemma we get

$$|\overline{\lim_{m\to\infty}} H_m| = 1.$$

If $x \in \overline{\lim_{m\to\infty}} H_m$ then the inequality (38) is satisfied for infinitely many values of m and hence

(44) $$\overline{\lim_{n\to\infty}} \left(\max_{N_n < i \leq N_n+\nu_n} \left| \sum_{k=N_n+1}^{i} c_k \varphi_k(x) \right| \right) = \infty$$

holds almost everywhere.

As to the Lebesgue functions

$$L_i(\{\varphi_k\}; x) = \int_0^1 \left| \sum_{k=0}^{i} \varphi_k(x)\varphi_k(t) \right| dt$$

of the system $\{\varphi_k(x)\}$ with $i=N_n$ and $i=N_n+\nu_n$, we have

$$L_{N_n}(\{\varphi_k\};x) \leqq 1+\sum_{r=1}^{n}\int_0^1\left|\sum_{k=N_{r-1}+1}^{N_r}\varphi_k(x)\varphi_k(t)\right|dt,$$

as $\varphi_0(x)\equiv 1$. From the definition of the intervals $I_i(n)$ $(i=1,2,3;\ n=2,3,\ldots)$, by (35) and (40), it follows

$$L_{N_n}(\{\varphi_k\};x) \leqq \begin{cases} C_{12} & \left(x\notin \bigcup_{l=2}^{\infty}(I_2(l)\cup I_3(l))\right), \\ C_{13}/\sqrt{|I_2(p)|} & (x\in I_2(p)), \\ C_{14} & (x\in I_3(q)) \quad (n=2,3,\ldots). \end{cases} \tag{45}$$

It follows exactly in the same way as before that

$$L_{N_n+\nu_n}(\{\varphi_k\};x) \leqq 1+\sum_{r=1}^{n}\int_0^1\left|\sum_{k=N_{r-1}+1}^{N_r}\varphi_k(x)\varphi_k(t)\right|dt+\int_0^1\left|\sum_{k=N_n+1}^{N_n+\nu_n}\varphi_k(x)\varphi_k(t)\right|dt,$$

and taking into consideration (35) and (39), we get the estimate

$$L_{N_n+\nu_n}(\{\varphi_k\};x) \leqq \begin{cases} C_{15} & \left(x\notin \bigcup_{l=2}^{\infty}(I_2(l)\cup I_3(l))\right), \\ C_{16}/\sqrt{|I_2(p)|} & (x\in I_2(p)), \\ C_{17}/\sqrt{|I_3(q)|} & (x\in I_3(q)) \quad (n=2,3,\ldots). \end{cases} \tag{46}$$

Hence and by (45) and (46), in virtue of (33), we obtain that

$$\int_0^1 (\sup_n L_{N_n}(\{\varphi_k\};x))\,dx<\infty, \qquad \int_0^1 (\sup_n L_{N_n+\nu_n}(\{\varphi_k\};x))\,dx<\infty.$$

Furthermore, (36) implies $\sum_{k=0}^{\infty} c_k^2<\infty$. Denote by $s_i(x)$ the i-th partial sum of the series (5). On account of a theorem of LEINDLER [5] it follows that $\{s_{N_n}(x)\}$ and $\{s_{N_n+\nu_n}(x)\}$ converge almost everywhere.

The above mentioned theorem of LEINDLER reads as follows:

Let $\{\varphi_k(x)\}$ $(k=0,1,\ldots)$ be an arbitrary orthonormal system in (a,b). If for a monotone increasing sequence $\{n_r\}$ of indices the inequality

$$L_{n_r}(\{\varphi_k\};x)=O(1) \qquad (a\leqq x\leqq b)$$

holds, then under the condition $\sum_{k=0}^{\infty} a_k^2<\infty$ the n_r-th partial sums of the orthogonal series (1) *converge almost everywhere.*

A more detailed analysis of LEINDLER's proof shows that the assertion remains valid under the weaker condition:

$$\sup_r L_{n_r}(\{\varphi_k\}; x) \in L(a, b).$$

Let us denote by $t_i(x)$ the i-th T-mean of the orthogonal series (5). If $N_n < i < N_n + \nu_n$ then on account of the definition of the matrix T and the sequence $\{c_k\}$, we have

$$t_i(x) = \tfrac{1}{2} s_i(x) + \tfrac{1}{2} s_{N_{n+1}-i}(x) = \tfrac{1}{2} s_{N_n}(x) + \tfrac{1}{2} \sum_{k=N_n+1}^{i} c_k \varphi_k(x) + \tfrac{1}{2} s_{N_n+\nu_n}(x).$$

Hence, if we pay attention to (44), it follows from the convergence of $\{s_{N_n}(x)\}$ and $\{s_{N_n+\nu_n}(x)\}$ that

$$\varlimsup_{i\to\infty} |t_i(x)| = \infty$$

almost everywhere. Thus the orthogonal series (5) is not T-summable almost everywhere in (0, 1).

To accomplish the proof of our theorem, we have to show that for the Lebesgue functions concerning the T-summation the relation (4) is satisfied.

If $N_n + \nu_n \leqq i \leqq N_{n+1}$ then

$$L_i(T; \{\varphi_k\}; x) = L_{N_{n+1}}(\{\varphi_k\}; x) \quad \text{and} \quad L_i(T; \{\varphi_k\}; x) = L_{N_n+\nu_n}(\{\varphi_k\}; x),$$

respectively, thus in virtue of (45) and (46) the following estimate

$$L_i(T; \{\varphi_k\}; x) \leqq \begin{cases} C_{18} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_{19}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{20}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{47}$$

$$(N_n + \nu_n \leqq i \leqq N_{n+1}; \quad n = 2, 3, \dots)$$

is true.

Finally, let $N_n < i < N_n + \nu_n$ be, i.e. $i = N_n + j$ $(1 \leqq j < \nu_n)$. Then

$$L_i(T; \{\varphi_k\}; x) = \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n+j} \varphi_k(x)\varphi_k(t) + \sum_{k=0}^{N_{n+1}-j} \varphi_k(x)\varphi_k(t) \right| dt.$$

A simple calculation shows

$$L_i(T; \{\varphi_k\}; x) \leqq \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n} \varphi_k(x)\varphi_k(t) \right| dt + \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n+\nu_n} \varphi_k(x)\varphi_k(t) \right| dt +$$

$$+ \tfrac{1}{2} \int_0^1 \left| \sum_{k=N_n+1}^{N_n+j} \varphi_k(x)\varphi_k(t) + \sum_{k=N_n+\nu_n+1}^{N_{n+1}-j} \varphi_k(x)\varphi_k(t) \right| dt = \tag{48}$$

$$= \tfrac{1}{2} \left(L_{N_n}(\{\varphi_k\}; x) + L_{N_n+\nu_n}(\{\varphi_k\}; x) + S_j(n; x) \right).$$

By virtue of (42) we get

$$S_j(n;x) \leqq \begin{cases} C_{11} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_9 L_{p^6}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{10} L_{q^6}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{49}$$

$$(1 \leqq j < \nu_n; \quad n = 2, 3, \ldots).$$

From the inequalities (45), (46), (48) and (49) it follows

$$L_i(T; \{\varphi_k\}; x) \leqq \begin{cases} C_{21} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_{22} L_{p^6}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{23} L_{q^6}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{50}$$

$$(N_n < i < N_n + \nu_n; \quad n = 2, 3, \ldots).$$

(Here we again took into consideration that $L_n \geqq 1$ for every n.) From (47) and (50) we infer that

$$\int_0^1 \sup_i L_i(T; \{\varphi_k\}; x)\, dx \leqq C_{24} \left(1 + \sum_{n=2}^{\infty} L_{n^6} \left(\sqrt{|I_2(n)|} + \sqrt{|I_3(n)|}\right)\right)$$

holds. Hence on account of (33) we obtain that (4) is fulfilled.

We have thus completed the proof of our theorem.

References

[1] G. Alexits, *Convergence Problems of Orthogonal Series* (Budapest, 1961).

[2] А. В. Ефимов, О несуммируемых линейными методами ортогональных рядах, *Доклады Акад. Наук СССР*, **152** (1963), 31-34.

[3] S. Kaczmarz, Sur la convergence et la sommabilité des développements orthogonaux, *Studia Math.*, **1** (1929), 87—121.

[4] L. Leindler, Über die sehr starke Riesz-summierbarkeit der Orthogonalreihen und Konvergenz lückenhafter Orthogonalreihen, *Acta Math. Acad. Sci. Hungaricae*, **13** (1962), 401—414.

[5] L. Leindler, Nicht verbesserbare Summierbarkeitsbedingungen für Orthogonalreihen, *Acta Math. Acad. Sci. Hungaricae*, **13** (1962), 425—432.

[6] Д. Е. Меньщов, О суммировании ортогональных рядов линейными методами, *Труды Москов. Мат. Общ.*, **10** (1961), 351—418.

[7] G. Sunouchi, On the Riesz summability of Fourier series, *Tôhoku Math. J.*, second series, **11** (1959), 319—326.

[8] K. Tandori, Über die Cesàrosche Summierbarkeit der orthogonalen Reihen, *Acta Sci. Math.* **14** (1951/52), 85—95.

[9] K. Tandori, Über die Divergenz der Walshschen Reihen, *Acta Sci. Math.*, **27** (1966), 261—263.

[10] K. Tandori, Ergänzung zu einem Satz von S. Kaczmarz, *Acta Sci. Math.*, **28** (1967), 147—153.

(Received October 25, 1967)

Berichtigung zur Arbeit „Über die starke Summation von Fourierreihen" *)

Von KÁROLY TANDORI in Szeged

Der Beweis des Satzes I dieser Arbeit ist falsch. Mit der dort angewandten Methode kann man nur die folgende, ziemlich komplizierte Behauptung beweisen:

Ist $f(t)$ nach 1 periodisch und in $[0, 1]$ *Lebesgue-integrierbar, so gibt es für fast alle Punkte* $x \in [0, 1]$ *eine positive Intervallfunktion* $\Phi_x(I)$ *mit* $\sum_{n=0}^{\infty} \Phi_x\left(\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right) < \infty$ *derart, daß für* $0 < k < \infty$ *und* $0 < h \to 0$ *gilt:*

$$(1) \qquad \int_h^{2h} |f(x+u)-f(x)|\,du \int_{u-k}^{u+k} |f(x+v)-f(x)|\,dv = o(h^2\,\Phi((h, 2h])) + o(hk),$$

undzwar gleichmäßig in Bezug auf k.

Ähnlicherweise, wie in der erwähnten Arbeit, kann bewiesen werden, daß aus (1) die H_2-Summierbarkeit der Fourierreihe von $f(t)$ in dem Punkt x folgt.

(Eingegangen am 28. März 1968)

*) *Acta Sci. Math.*, **16** (1955), 65—73.