On the cosine of unbounded operators

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The concept of the cosine $\cos_R A$ of an accretive linear operator A is a useful parameter in determining when the product of two accretive operators is itself accretive (see [2]). To be (nontrivially) applicable in that context, it is necessary that the cosine be strictly positive, which is always the case for strongly accretive bounded operators. The object of the present note is to show that $\cos_R A \equiv 0$ for all unbounded accretive operators A. This fact seems interesting since it serves to distinguish geometrically the topological notions of boundedness and unboundedness for strongly accretive operators.

We restrict ourselves to (real or complex, separable or non-separable) pre-Hilbert spaces H and to unbounded accretive operators A. It is not necessary that A be closed or densely defined, and no completeness properties for H are needed; however, it should be noted that when H is a Hilbert space and $D(A)$, the domain of A, is dense, then the accretiveness of A implies that A is closeable (see [3, p. 268]). Our demonstration does not immediately extend to Banach spaces because we make use of both bilinearity of the inner product (not generally present for the semi-inner product, see [1, 4]) and orthogonality in H.

We recall that an operator A is accretive if $\operatorname{Re}(Ax, x) \geqq 0$ for all $x \in D(A)$.

Definition.

$$\cos_R A = \inf_x \frac{\operatorname{Re}(Ax, x)}{\|Ax\| \cdot \|x\|}, \quad |\cos| A = \inf_x \frac{|(Ax, x)|}{\|Ax\| \cdot \|x\|} \qquad (x \in D(A),\ Ax \neq 0).$$

Theorem. $\cos_R A = 0$ *for all accretive unbounded operators* A.

Proof. If $\operatorname{Re}(Ax, x)$ is bounded above uniformly, $\cos A = 0$ immediately by the unboundedness of A; therefore we may assume that there exists a sequence $\{u_n\}, \|u_n\| = 1$, $\operatorname{Re}(Au_n, u_n) \to \infty$. Let $w_n = \eta_n u_n + \xi_n v_n$ where $\eta_n = [\operatorname{Re}(Au_n, u_n)]^{-\alpha}$, $\xi_n = (1 - \eta_n^2)^{\frac{1}{2}}$, $\frac{1}{2} \leqq \alpha < 1$, and $v_n \in D(A)$, $\|v_n\| = 1$; v_n will be specifically chosen

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later. Then for all sufficiently large n, if $\|Av_n\|$ is uniformly bounded, one has by the inverse triangle inequality:

$$R(w_n) = \frac{\operatorname{Re}(Aw_n, w_n)}{\|Aw_n\| \cdot \|w_n\|} \leqq \tag{1}$$

$$\leqq \frac{\xi_n^2 \operatorname{Re}(Av_n, v_n) + \xi_n \eta_n \operatorname{Re}(Av_n, u_n) + \eta_n \xi_n \operatorname{Re}(Au_n, v_n) + \eta_n^2 \operatorname{Re}(Au_n, u_n)}{[\eta_n \|Au_n\| - \xi_n \|Av_n\|] \cdot |\xi_n - \eta_n|} =$$

$$= (N_1 + N_2 + N_3 + N_4)/D,$$

with the denominator $D \to \infty$, since $|\xi_n - \eta_n| \to 1$, and $\|Au_n\| \cdot [\operatorname{Re}(Au_n, u_n)]^{-\alpha} \to \infty$ for $0 < \alpha < 1$; the latter may be seen as follows. Let $\|u_n\| = 1, \alpha < 1$, and $\operatorname{Re}(Au_n, u_n) \to \infty$. Then by SCHWARZ's inequality one has $\|Au_n\|^2 \cdot [\operatorname{Re}(Au_n, u_n)]^{-2\alpha} \geqq [\operatorname{Re}(Au_n, u_n)]^{2-2\alpha} \to \infty$.

Let us now consider the four terms N_i/D separately. If $\|Av_n\|$ is uniformly bounded, clearly (by SCHWARZ's inequality) $N_1/D \to 0$ and $N_2 \to 0$. Also, $N_4 = 1$ if $\alpha = 1/2$, $N_4 \to 0$ if $\alpha > 1/2$; thus $(N_1 + N_2 + N_4) \cdot D^{-1} \to 0$ for $1/2 \leqq \alpha < 1$. Therefore if $|N_3|$ is uniformly bounded, $R(w_n) \to 0$ in (1). Now, if there exists at least one nontrivial vector $v \in D(A) \cap D(A^*)$ (let it have norm $= 1$), then taking $v_n \equiv v$, $\|Av_n\|$ is obviously uniformly bounded, and $|N_3| = \eta_n \xi_n |\operatorname{Re}(u_n, A^* v_n)| \leqq \|A^* v\|$. If $D(A) \cap D(A^*) = \{0\}$, we may proceed as follows. Select $x, y \in D(A)$, $\|x\| = \|y\| = 1$, $(x, y) = 0$, and let $v_n = \alpha_n x + \beta_n y$, $|\alpha_n|^2 + |\beta_n|^2 = 1$. Now choose α_n, β_n so that $(v_n, Au_n) = 0$; that this can always be done is assured by taking α_n and β_n from the solutions of the equation $\alpha_n(x, Au_n) + \beta_n(y, Au_n) = 0$. Then $\|v_n\| = 1$, $\|Av_n\| \leqq \|Ax\| + \|Ay\|$, $N_3 = 0$, and $R(w_n) \to 0$.

One may obtain the following stronger result.

Corollary. $|\cos| A = 0$ *for all unbounded operators.*

Proof. Replace $\operatorname{Re}(Ax_1, x_2)$ by $|(Ax_1, x_2)|$ everywhere in the above.

References

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