A note on invariant linear forms on von Neumann algebras

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1. Let $\mathbf{A}$ be a von Neumann algebra [1]) in a complex Hilbert space $\mathfrak{H}$. We shall denote by $\mathbf{A}^+$ (resp. $\mathbf{A}_1$) the positive portion (resp. the unit ball) of $\mathbf{A}$: $\mathbf{A}^+ = \{T \in \mathbf{A}: T \geqq O\}$ (resp. $\mathbf{A}_1 = \{T \in \mathbf{A}: \|T\| \leqq 1\}$. If $\mathcal{G}$ is a group of $*$-automorphisms of $\mathbf{A}$, set $\mathbf{A}^{\mathcal{G}} = \{T \in \mathbf{A}: \theta(T) = T$ for each $\theta \in \mathcal{G}\}$. It is not hard to see that $\mathbf{A}^{\mathcal{G}}$ is a von Neumann subalgebra of $\mathbf{A}$. For a given pair $\mathbf{A}, \mathcal{G}$, we shall denote by $\mathcal{R}(\mathbf{A}, \mathcal{G})$ the space of all ultra-weakly continuous linear forms σ on $\mathbf{A}$ which are invariant with respect to $\mathcal{G}$. [2]) $\mathcal{R}^r(\mathbf{A}, \mathcal{G})$ (resp. $\mathcal{R}^+(\mathbf{A}, \mathcal{G})$) will denote the set of all real (resp. positive) [3]) elements of $\mathcal{R}(\mathbf{A}, \mathcal{G})$.

Consider an arbitrary element σ of $\mathcal{R}^r(\mathbf{A}, \mathcal{G})$ and set $M = \operatorname{l.u.b.}_{T \in \mathbf{A}_1^+} \sigma(T)$, where $A_1^+ = \mathbf{A}_1 \cap \mathbf{A}^+$. According to [4], there exists a projection E in $\mathbf{A}_1^+$ such that $\sigma(E) = M$. In this note we intend first to prove that E can be chosen to be an element of $\mathbf{A}^{\mathcal{G}}$. Then we shall study some consequences of this fact. Let us formulate it in the form of a

Theorem. *Let σ be an arbitrary element of $\mathcal{R}^r(\mathbf{A}, \mathcal{G})$ and set $M = \operatorname{l.u.b.}_{T \in \mathbf{A}_1^+} \sigma(T)$. Then there exists a projection F of $\mathbf{A}^{\mathcal{G}}$ such that $\sigma(F) = M$.*

Remark. If $\mathcal{G}$ is abelian, the theorem is a consequence of the Kakutani—Markov fixed point theorem (cf. [2, V. 10. 6]). In case A is G-finite ([3]), the support [4]) of $E^{\mathcal{G}}$ has the property required (cf. [4]).

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1) For the theory of von Neumann algebras we refer the reader to [1].

2) I. e. $\sigma(\theta(T)) = \sigma(T)$ for every $T \in \mathbf{A}$ and $\theta \in \mathcal{G}$.

3) For any $\sigma \in \mathcal{R}(\mathbf{A}, \mathcal{G})$, put $\sigma^*(T) = \overline{\sigma(T^*)}$. It is evident that $\sigma^* \in \mathcal{R}(\mathbf{A}, \mathcal{G})$. σ is said to be *real* (resp. *positive*) if $\sigma = \sigma^*$ (resp. $\sigma(T) \geqq 0$ for $T \in \mathbf{A}^+$). Any $\sigma \in \mathcal{R}(\mathbf{A}, \mathcal{G})$ can be uniquely written in the form $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_1, \sigma_2 \in \mathcal{R}^r(\mathbf{A}, \mathcal{G})$. In fact, $\sigma_1 = \frac{1}{2}(\sigma + \sigma^*)$ and $\sigma_2 = \frac{1}{2i}(\sigma - \sigma^*)$. In the following we prefer to use the notations: $\sigma_1 = \operatorname{Re} \sigma$, $\sigma_2 = \operatorname{Im} \sigma$.

4) Cf. [1, App. III].

Proof of the theorem. Actually, the proof is based upon the following simple fact: Let $T_1, \cdots, T_n$ $(n=1, 2, \cdots)$ be elements in $\mathbf{A}^+$ with supports $E_1, \cdots, E_n$, respectively. Then the support E of $T = T_1 + \cdots + T_n$ is equal to $\underset{1 \leq i \leq n}{\text{l.u.b.}}\, E_i$. It is enough to show this statement for $n=2$, since then an induction argument will take care of the general case. For $n=2$ we prove that $I-E=(I-E_1)\cap(I-E_2)$ [5]) which evidently implies the assertion. Now the inequality $(I-E_1)\cap(I-E_2) \leqq I-E$ is evident. To prove the converse of it, consider an arbitrary element x of $(I-E)\mathfrak{H}$. Then $Tx=(T_1+T_2)x=0$. Thus, $(T_1+T_2x|x)=0$ which gives that $(T_1x|x) = -(T_2x|x)$. As $T_1, T_2 \in \mathbf{A}^+$, this is possible only if $(T_1x|x)=(T_2x|x)=0$. But this implies that $T_1^{\frac{1}{2}}x = T_2^{\frac{1}{2}}x = 0$, i.e. $T_1x=T_2x=0$. Hence $I-E \leqq (I-E_1) \cap (I-E_2)$, which was to be proved. Now let $E \in \mathbf{A}_1^+$ be a projection such that $\sigma(E)=M$ (cf. [4]), and consider an arbitrary finite system $\theta_1, \cdots, \theta_n$ of elements of $\mathcal{G}$. Set $S_n = \frac{1}{n}\sum_{i=1}^{n}\theta_i(E)$. It is evident that $S_n \in \mathbf{A}_1^+$ and $\sigma(S_n) = \frac{1}{n}\sum_{i=1}^{n}\sigma(\theta_i(E)) = \sigma(E) = M$. Since, for every i $(1 \leqq i \leqq n)$, $\theta_i(E)$ is a projection, the support F_n of S_n is equal to $\underset{1 \leq i \leq n}{\text{l.u.b.}}\, \theta_i(E)$. Since $\sigma(S_n)=M$, we obtain that $\sigma(F_n)=M$, too (cf. [4]). Now if for each possible finite system J of elements of $\mathcal{G}$ we take the corresponding projection F_J, we obtain an upward directed family $\{F_J\}_{J \subset \mathcal{G}}$ of projections. Put $F = \underset{J \subset \mathcal{G}}{\text{l.u.b.}}\, F_J$. F is a cluster point of the projections F_J in the strong operator topology (cf. [1, App. III]). Now taking into account the topological properties of the elements of $\mathcal{G}$ and σ (cf. [1, chap. I. §§ 3—4]), for every $\theta_0 \in \mathcal{G}$ we have $\theta_0(F) = \theta_0(\underset{J \subset \mathcal{G}}{\text{l.u.b.}}\, F_J) = \underset{J \subset \mathcal{G}}{\text{l.u.b.}}\, \theta_0(F_J) = \underset{J' \subset \mathcal{G}}{\text{l.u.b.}}\, F_{J'}$ [6]) $= F$, i.e. $F \in \mathbf{A}^{\mathcal{G}}$, and $\sigma(F) = \underset{J \subset \mathcal{G}}{\text{l.u.b.}}\, \sigma(F_J)$. But, by construction, $\sigma(F_J)=M$ for each $J(\subset \mathcal{G})$, thus $\sigma(F)=M$, qu.e.d.

2. Now let us present some consequences of this theorem.

Let σ_1 and σ_2 be two arbitrary elements of $\mathcal{R}(\mathbf{A}, \mathcal{G})$ such that for every $T \in \mathbf{A}^{\mathcal{G}}$ we have $\sigma_1(T)=\sigma_2(T)$. Put $\sigma=\sigma_1-\sigma_2$, $\sigma'=\operatorname{Re}\sigma$ and $\sigma''=\operatorname{Im}\sigma$. Since $\sigma'(T) = \sigma''(T)=0$ for every $T \in \mathbf{A}^{\mathcal{G}}$, by the preceding theorem we obtain that $\underset{S \in \mathbf{A}_1^+}{\text{l.u.b.}}\, \sigma'(S) = \underset{S \in \mathbf{A}_1^+}{\text{l.u.b.}}\, \sigma''(S)=0$. Arguing with $-\sigma$ instead of σ, we can see that $\underset{S \in \mathbf{A}_1^+}{\text{g.l.b.}}\, \sigma'(S) = \underset{S \in \mathbf{A}_1^+}{\text{g.l.b.}}\, \sigma''(S)=0$ as well. This means that $\sigma'=\sigma''=0$, i.e. $\sigma=0$, hence $\sigma_1=\sigma_2$. [7])

Thus we obtain

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[5]) For two projections P and Q, $P \cap Q$ denotes the projection of $\mathfrak{H}$ onto the subspace $(P\mathfrak{H}) \cap (Q\mathfrak{H})$.

[6]) J' also runs over all possible finite systems of elements of G.

[7]) For $\sigma_1, \sigma_2 \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$, this fact has been already established in [3].

Proposition 1. *Each element of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is uniquely determined by its restriction to $\mathbf{A}^{\mathscr{G}}$.*

Suppose now that σ belongs to $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$, and denote by F the projection in $\mathbf{A}^{\mathscr{G}}$ such that $\sigma(F) = \underset{T \in \mathbf{A}_I^+}{\text{l.u.b.}}\, \sigma(T)$ (cf. Theorem). Put $\sigma_1(T) = \sigma(FT)$ and $\sigma_2(T) = -\sigma\big((I-F)T\big)$ $(T \in \mathbf{A})$. Then, according to [4], σ_1 and σ_2 are ultra-weakly continuous positive linear forms on $\mathbf{A}$ with disjoint supports (cf. [1, chap. I, § 4, no. 6]) such that $\sigma = \sigma_1 - \sigma_2$. Moreover, it is evident that $\sigma_1, \sigma_2 \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Thus we have

Proposition 2. *Each element of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ can be represented as a difference of two elements of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with disjoint supports. Therefore, each element of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is a finite linear combination of elements of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$. If $\mathbf{A}^{\mathscr{G}}$ reduces to the trivial von Neumann algebra of the scalars, then each element of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ is either positive or negative* (i.e. a positive linear form multiplied by -1).

From this proposition we can conclude at once the following

Proposition 3. *$\mathbf{A}$ is $\mathscr{G}$-finite if and only if for every $T \in \mathbf{A}^+$, $T \neq 0$, there exists an element σ of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ such that $\sigma(T) \neq 0$.*

In particular if $\mathscr{G}$ is the group of all inner automorphisms of $\mathbf{A}$, then $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is identical with the space of all ultra-weakly continuous central (cf. [1, p. 275]) linear forms on $\mathbf{A}$. In this case Propositions 2 and 3 assert that

(i) *Each ultra-weakly continuous central linear form on $\mathbf{A}$ is a linear combination of finite normal traces* (cf. [1, chap. I. § 6, def. 1]). *In particular if $\mathbf{A}$ is a factor, and $\sigma \neq 0$ is an ultra-weakly continuous central linear form on $\mathbf{A}$, then $\mathbf{A}$ is finite and σ is a scalar multiple of the canonical trace of $\mathbf{A}$. Furthermore if $\mathbf{A}$ is a properly infinite von Neumann algebra* (cf. [1. chap. I, § 6. def. 5]) *then there exists no non-zero ultra-weakly continuous central linear form on $\mathbf{A}$.*

(ii) *$\mathbf{A}$ is finite if and only if for every $T \in \mathbf{A}^+$, $T \neq 0$ there exists an ultra-weakly continuous central linear form σ on $\mathbf{A}$ such that $\sigma(T) \neq 0$.*

Bibliography

[1] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1957).
[2] N. DUNFORD and J. T. SCHWARTZ, *Linear operators.* I (New York, 1958).
[3] I. KOVÁCS and J. SZŰCS, Ergodic-type theorems in von Neumann algebras, *Acta Sci. Math.*, **27** (1966), 233—246.
[4] L. PUKÁNSZKY, The theorem of Radon—Nikodym in operator-rings, *Acta Sci. Math.*, **15** (1953—54), 149-156.

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