On the T-summation of orthogonal series

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Introduction

Let $\{\varphi_n(x)\}_1^\infty$ be an arbitrary orthonormal system (in abbreviation "ONS") in $[0, 1]$. We shall consider series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{1}$$

with real coefficients, $\{c_n\} \in l^2$. By the Riesz—Fischer theorem, (1) converges in the mean to a square integrable function $f(x)$.

Let B be the class of those $\{c_n\}_1^\infty$ for which (1) converges almost everywhere (in abbreviation "a.e.") for every ONS in $[0, 1]$. (The set of divergence points may depend on the system $\{\varphi_n(x)\}$.) TANDORI [3] proved the following

Theorem. *For any sequence* $\mathfrak{c} = \{c_n\}_1^\infty$ *of real numbers set*

$$I(c_1, \dots, c_N) = \sup \int_0^1 \Big(\max_{1 \le i \le j \le N} |c_i \varphi_i(x) + \dots + c_j \varphi_j(x)| \Big)^2 dx,$$

the supremum being taken over all ONS in $[0, 1]$; *furthermore, define*

$$\|\mathfrak{c}\| = \lim_{N \to \infty} I^{1/2}(c_1, \dots, c_N) \qquad (\le \infty).$$

We have $\mathfrak{c} \in B$ *if and only if* $\|\mathfrak{c}\| < \infty$. *B is a Banach space with respect to the usual vector operations and the norm* $\|\mathfrak{c}\|$.

The aim of this paper is to extend this result to T-summability instead of convergence. More exactly, let $T = (a_{ik})_{i,k=1}^\infty$ be a double infinite matrix of real numbers satisfying the conditions

$$\lim_{i \to \infty} a_{ik} = 0 \qquad (k = 1, 2, \dots), \tag{2}$$

$$\lim_{i \to \infty} \sum_{k=1}^{\infty} a_{ik} = 1, \tag{3}$$

and

(4) $$\sum_{k=1}^{\infty} |a_{ik}| \leqq K \qquad (i = 1, 2, \ldots),$$ [1])

where K is a positive constant. In the sequel, we use $K, K_1, K_2, \ldots$ to denote positive constants. Set

$$A_{in} = \sum_{k=n}^{\infty} a_{ik} \qquad (n = 1, 2, \ldots).$$

We denote by $s_k(x)$ the kth partial sum of (1). The series (1) is called *T-summable* at the point $x \in [0, 1]$ if

$$t_i(x) = \sum_{k=1}^{\infty} a_{ik} s_k(x) = \sum_{n=1}^{\infty} A_{in} c_n \varphi_n(x)$$

exists for all i, and

$$\lim_{i \to \infty} t_i(x) = f(x).$$

Let $B(T)$ be the class of those $\{c_n\}_1^\infty$ for which (1) is T-summable a.e. for every ONS in [0, 1]. (The set, in the points of which (1) is not T-summable, depends on the system $\{\varphi_n(x)\}$.) We note that if (1) is T-summable a.e. for every ONS in [0, 1], then necessarily $\{c_n\} \in l^2$. For example, the Rademacher series $\sum c_n r_n(x)$ is not T-summable when $\sum c_n^2 = \infty$. (See ZYGMUND [5].) Hence we infer that $B(T) \subsetneqq l^2$.

Our principal result is the following

Theorem 1. *Let T be a matrix satisfying conditions* (2), (3) *and* (4). *For any sequence $\mathfrak{c} = \{c_n\}_1^\infty$ of real numbers set*

$$I(T, \mathfrak{c}, N) = \sup \int_0^1 (\max_{1 \leqq i \leqq N} |t_i(x)|)^2 \, dx,$$ [2])

the supremum being taken over all ONS *in* [0, 1]; *furthermore, define*

(5) $$\|\mathfrak{c}\|_T = \lim_{N \to \infty} I^{1/2}(T, \mathfrak{c}, N) \qquad (\leqq \infty).$$

We have $\mathfrak{c} \in B(T)$ if and only if $\|\mathfrak{c}\|_T < \infty$. $B(T)$ is a Banach space with respect to the usual vector operations and the norm $\|\mathfrak{c}\|_T$.

In a number of important special cases such as (C, α)-summability or $(R, \lambda_n, 1)$-summability (see ALEXITS [1], p. 139) there exists an increasing sequence $\mathfrak{n} = \{n_i\}$

[1]) We note that the conditions (2)—(4) are necessary and sufficient for the permanence of the T-summation. (See ZYGMUND [4], p. 74.)

[2]) This is evidently a non-decreasing function of N.

of natural numbers such that, under $\mathfrak{c} \in l^2$, the a.e. T-summability of (1) for every ONS is equivalent to the a.e. convergence of the sequence of the n_ith partial sums of (1). In this special case, we have $B(T)=B(T_\mathfrak{n})$, where $T_\mathfrak{n}$ is defined as follows: for every i put $a_{i,n_i}=1$ and $a_{ik}=0$ if $k \neq n_i$; then our Theorem 1 includes Theorem II of TANDORI [3] as a particular case. We note that, as MENCHOFF [2] showed, there exists a matrix T with (2), (3) and (4) such that for any increasing sequence $\mathfrak{n}$ of natural numbers we have $B(T) \neq B(T_\mathfrak{n})$.

The following theorems are the extensions of those of TANDORI that can also be found in his cited paper.

We say that (1) is "*boundedly*" T-summable if

(i) it is T-summable a.e. in $[0, 1]$;

(ii) the T-means $t_i(x)$ are majorized by some square integrable function, the square integral of which has a bound depending only on the sequence $\mathfrak{c}$ of coefficients.

Theorem 2. *The a.e. T-summability of the series* (1) *for every* ONS *is equivalent to its bounded T-summability for every* ONS *in* $[0, 1]$.

The following three theorems contain assertions concerning some properties of the norm $\|\mathfrak{c}\|_T$ and of the class $B(T)$.

Theorem 3. *Let $\mathfrak{c}=\{c_n\}_1^\infty$ and $\mathfrak{d}=\{d_n\}_1^\infty$ be two sequences of real numbers with $|c_n| \leqq |d_n|$ $(n=1, 2, \cdots)$. If $\mathfrak{d} \in B(T)$ then $\mathfrak{c} \in B(T)$ and $\|\mathfrak{c}\|_T \leqq \|\mathfrak{d}\|_T$.*

Theorem 4. *Let $\mathfrak{c}_m=\{c_{mn}\}_{n=1}^\infty$ $(m=1, 2, \cdots)$ be such that, for every fixed n, c_{mn} is a decreasing sequence in m and tends to* 0. *Suppose, moreover, that $\|\mathfrak{c}_1\|_T<\infty$. Then $\|\mathfrak{c}_m\|_T \to 0$ $(m \to \infty)$.*

Theorem 5. *$B(T)$ is separable.*

Finally we note, without any proof, that Theorem 1 remains valid if (5) is replaced by

$$\|\mathfrak{c}\|_T^{(p)} = \lim_{N\to\infty} I_p^{1/p}(T, \mathfrak{c}, N) \qquad (1 \leqq p \leqq 2),$$

where

$$I_p(T, \mathfrak{c}, N) = \sup \int_0^1 \big(\max_{1 \leqq i \leqq N} |t_i(x)|\big)^p \, dx,$$

the supremum being taken over all ONS in $[0,1]$.

§1. Lemmas

The proofs of the theorems depend on several lemmas. First, let us introduce the quantity

$$J(\mathfrak{c}, M, N) = J(T, \mathfrak{c}, M, N) = \sup \int_0^1 \big(\max_{M \leqq i < j \leqq N} |t_j(x) - t_i(x)| \big)^2 \, dx =$$

$$= \sup \int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\infty} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx,$$

where M and N denote natural numbers with $M < N$, and the supremum is taken over all ONS in $[0, 1]$. Sometimes, if it does not cause any misunderstanding, instead of $I(T, \mathfrak{c}, N)$, $\|\mathfrak{c}\|_T$ and $B(T)$ we shall write $I(\mathfrak{c}, N)$, $\|\mathfrak{c}\|$ and B, respectively. It is obvious that

$$\frac{1}{2} I(\mathfrak{c}, N) - I(\mathfrak{c}, M) \leqq J(\mathfrak{c}, M, N) \leqq 4I(\mathfrak{c}, N). \tag{6}$$

In the sequel we shall work with the projections P_μ, P^ν and P_μ^ν defined as follows: for any given $\mathfrak{c} = \{c_n\}_1^\infty$ we denote by $P_\mu \mathfrak{c}$ the sequence that comes from $\mathfrak{c}$ by replacing the first $\mu - 1$ components of $\mathfrak{c}$ with 0, that is $P_\mu \mathfrak{c} = \{0, \cdots, 0, c_\mu, c_{\mu+1}, \cdots\}$; similarly, $P^\nu \mathfrak{c} = \{c_1, \cdots, c_\nu, 0, 0, \cdots\}$; and $P_\mu^\nu \mathfrak{c} = P_\mu P^\nu \mathfrak{c} = \{0, \cdots, 0, c_\mu, c_{\mu+1}, \cdots, c_\nu, 0, 0, \cdots\}$ $(1 \leqq \mu \leqq \nu)$.

In the following lemmas we always suppose that $\mathfrak{c} \in l^2$.

Lemma 1. *Let ε be a positive real number. Then there exists a natural number $N_0 = N_0(\varepsilon)$ such that*

$$I(\mathfrak{c}, N) \geqq (1 - \varepsilon) \sum_{n=1}^{\infty} c_n^2 - \varepsilon \tag{7}$$

holds for every $N \geqq N_0$; furthermore, for every natural number N and ν, we have

$$I(T^\nu \mathfrak{c}, N) \leqq K^2 \left(\sum_{n=1}^{\nu} |c_n| \right)^2. \tag{8}$$

Proof. To prove (7) we start with the relations

$$I(\mathfrak{c}, N) \geqq \int_0^1 t_N^2(x) \, dx = \sum_{n=1}^{\infty} A_{N,n}^2 c_n^2. \tag{9}$$

Because $\mathfrak{c} \in l^2$ we can fix the natural number $\nu_0 = \nu_0(\varepsilon)$ such that

$$\sum_{n=\nu_0+1}^{\infty} c_n^2 < \varepsilon.$$

By virtue of (2) and (3), there exists a natural number $N_0 = N_0(\varepsilon)$ such that for every $l \geqq \nu_0$ we have

$$|A_{Nl}^2 - 1| \leqq 2\left|\left(\sum_{k=1}^{\infty} a_{Nk}\right)^2 - 1\right| + 2\left(\sum_{k=1}^{l} a_{Nk}\right)^2 \leqq \varepsilon \quad \text{if} \quad N \geqq N_0.$$

By (9) we get

$$I(\mathfrak{c}, N) \geqq (1-\varepsilon) \sum_{n=1}^{\nu_0} c_n^2 \geqq (1-\varepsilon)\left(\sum_{n=1}^{\infty} c_n^2 - \varepsilon\right) \quad \text{if} \quad N \geqq N_0.$$

As to (8), it is sufficient to consider the following inequality:

$$\max_{1 \leqq i \leqq N} \left|\sum_{n=1}^{\nu} A_{in} c_n \varphi_n(x)\right| \leqq K^2 \sum_{n=1}^{\nu} |c_n \varphi_n(x)|.$$

Here we took (4) into consideration.

The proof of Lemma 1 is complete.

Lemma 2. *Let λ, μ, ν and N be natural numbers, $\lambda < \mu < \nu \leqq \infty$. Then we have*

$$I(P_\lambda^\mu \mathfrak{c}, N) + I(P_{\mu+1}^\nu \mathfrak{c}, N) \leqq I(P_\lambda^\nu \mathfrak{c}, N),$$

and in particular

$$I(P^\nu \mathfrak{c}, N) \leqq I(\mathfrak{c}, N).$$

The proof of Lemma 2 is analogous to that of Lemma IV of Tandori [3].

Lemma 3. *Let M and N be natural numbers, $M < N$, and let ε be a positive real number. Then there exists a natural number $\nu_0 = \nu_0(M, N, \varepsilon)$ such that*

(10) $$J(P_{\nu+1} \mathfrak{c}, M, N) \leqq \varepsilon,$$

and

$$J(P^\nu \mathfrak{c}, M, N) \geqq J(\mathfrak{c}, M, N) - \varepsilon$$

hold for every $\nu \geqq \nu_0$. The similar assertions concerning $I(\mathfrak{c}, N)$ are also valid.

Proof. It is sufficient to prove (10), as the second inequality is a simple consequence of (10), e.g. using the inequality $(a+b)^2 \geqq a^2 - 2|a|\,|b|$. Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ in [0, 1]. It is clear that

$$\left(\max_{M \leqq i < j \leqq N} \left|\sum_{n=\nu+1}^{\infty} (A_{jn} - A_{in}) c_n \varphi_n(x)\right|\right)^2 \leqq 4 \sum_{i=M}^{N} \left(\sum_{n=\nu+1}^{\infty} A_{in} c_n \varphi_n(x)\right)^2.$$

Integrating over [0, 1] term by term, on account of (4) we get

$$\int_0^1 \left(\max_{M \leqq i < j \leqq N} \left|\sum_{n=\nu+1}^{\infty} (A_{jn} - A_{in}) c_n \varphi_n(x)\right|\right)^2 dx \leqq 4(N-M+1)K^2 \sum_{n=\nu+1}^{\infty} c_n^2 < \varepsilon,$$

if ν is large enough, since $\mathfrak{c} \in l^2$. Since this estimate is valid for every ONS in [0, 1], we obtain (10).

This finishes the proof of Lemma 3.

Lemma 4. *Let μ be a natural number and let ε be a positive real number. Then there exists $M_0=M_0(\mu,\varepsilon)$ such that*

$$J(P^{\mu}\mathfrak{c}, M, N) \leqq \varepsilon, \tag{11}$$

and

$$J(P_{\mu+1}\mathfrak{c}, M, N) \geqq J(\mathfrak{c}, M, N)-\varepsilon$$

hold whenever $M_0 \leqq M < N$.

Proof. It is also sufficient to prove (11). Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ in $[0, 1]$. By a simple calculation we get

$$\int_0^1 \left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=1}^{\mu}(A_{jn}-A_{in})c_n\varphi_n(x)\right|\right)^2 dx \leqq$$

$$\leqq \left(\max_{M\leqq i<j\leqq N}|A_{jn}-A_{in}|\right)^2 \int_0^1\left(\sum_{n=1}^{\mu}|c_n\varphi_n(x)|\right)^2 dx \leqq \left(\sup_{M\leqq i<j}|A_{jn}-A_{in}|\right)^2\left(\sum_{n=1}^{\mu}|c_n|\right)^2.$$

By virtue of (2) and (3), there exists a natural number M_0 such that for every $n\leqq\mu$ we have

$$(A_{jn}-A_{in})^2 \leqq 4\left\{\left(\sum_{k=1}^{\infty}a_{jk}-1\right)^2+\left(\sum_{k=1}^{\infty}a_{ik}-1\right)^2+\left(\sum_{k=1}^{\mu}a_{jk}\right)^2+\left(\sum_{k=1}^{\mu}a_{ik}\right)^2\right\} \leqq \frac{\varepsilon}{\left(\sum_{n=1}^{\mu}|c_n|\right)^2}$$

whenever $M_0\leqq i<j$, whence

$$\int_0^1 \left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=1}^{\mu}(A_{jn}-A_{in})c_n\varphi_n(x)\right|\right)^2 dx \leqq \varepsilon \quad \text{if} \quad M_0\leqq M<N.$$

Since this is valid for every ONS in $[0, 1]$, (11) follows.

Thus the proof is complete.

Lemma 5. *The inequality*

$$I^{1/2}(\mathfrak{c}+\delta, N) \leqq I^{1/2}(\mathfrak{c}, N)+I^{1/2}(\delta, N)$$

holds.

Lemma 6. *Let L, M and N be natural numbers, $L<M<N$. Then the inequalities*

$$J^{1/2}(\mathfrak{c}, L, N) \leqq J^{1/2}(\mathfrak{c}, L, M)+J^{1/2}(\mathfrak{c}, M, N),$$

and

$$I^{1/2}(\mathfrak{c}, N) \leqq I^{1/2}(\mathfrak{c}, M)+J^{1/2}(\mathfrak{c}, M, N)$$

hold.

The proofs of Lemma 5 and Lemma 6 are similar to that of Lemma II of Tandori [3].

Lemma 7. *Let ν and N be natural numbers. Then $I(P^{\nu}\mathfrak{c}, N)$ is a continuous function of the coefficients c_n.*

Proof. This is an immediate consequence of Lemma 1 and Lemma 5.

Lemma 8. *Let M_1 and N_1 be natural numbers, $M_1 < N_1$, and let ε be a positive real number. Then there exists a natural number $M_0 = M_0(M_1, N_1, \varepsilon) > N_1$ such that*

$$J(\mathfrak{c}, M_1, N_1) + J(\mathfrak{c}, M_2, N_2) \leqq J(\mathfrak{c}, M_1, N_2) + \varepsilon,$$

whenever $M_0 \leqq M_2 < N_2$.

Proof. By virtue of Lemma 3 there exists a natural number $\nu_0 = \nu_0(M_1, N_1, \varepsilon)$ such that

$$J(P^{\nu_0}\mathfrak{c}, M_1, N_1) \geqq J(\mathfrak{c}, M_1, N_1) - \frac{\varepsilon}{4}.$$

According to Lemma 4 there exists a natural number $M_0 = M_0(\nu_0, \varepsilon) = M_0(M_1, N_1, \varepsilon)$ such that

$$J(P_{\nu_0+1}\mathfrak{c}, M_2, N_2) \geqq J(\mathfrak{c}, M_2, N_2) - \frac{\varepsilon}{4},$$

whenever $M_0 \leqq M_2 < N_2$. Thus there exist ONS $\{\varphi_n(x)\}_1^{\nu_0}$ and $\{\psi_n(x)\}_{\nu_0+1}^{\infty}$ in $[0, 1]$ for which

$$\begin{aligned} &\int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \geqq J(\mathfrak{c}, M_1, N_1) - \frac{\varepsilon}{2}, \\ (12) \quad &\int_0^1 \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(x) \right| \right)^2 dx \geqq J(\mathfrak{c}, M_2, N_2) - \frac{\varepsilon}{2}. \end{aligned}$$

Set, for $n = 1, 2, \ldots, \nu_0$,

$$\chi_n(x) = \sqrt{2}\, \varphi_n(2x) \quad \text{if} \quad 0 \leqq x \leqq \frac{1}{2}, \text{ and } \chi_n(x) = 0 \text{ otherwise;}$$

and, for $n = \nu_0 + 1, \nu_0 + 2, \ldots$,

$$\chi_n(x) = \sqrt{2}\, \psi_n(2x - 1) \quad \text{if} \quad \frac{1}{2} < x \leqq 1, \text{ and } \chi_n(x) = 0 \text{ otherwise.}$$

It is obvious that $\{\chi_n(x)\}_1^{\infty}$ is an ONS in $[0, 1]$, and it follows from (12) that

$$\begin{aligned} &J(\mathfrak{c}, M_1, N_1) + J(\mathfrak{c}, M_2, N_2) - \varepsilon \leqq \\ &\leqq \int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx + \\ &+ \int_0^1 \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(x) \right| \right)^2 dx = \end{aligned}$$

$$= 2\int_0^{1/2} \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(2x) \right| \right)^2 dx +$$

$$+ 2\int_{1/2}^{1} \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(2x-1) \right| \right)^2 dx \leqq$$

$$\leqq \int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_2} \left| \sum_{n=1}^{\infty} (A_{jn} - A_{in}) c_n \chi_n(x) \right| \right)^2 dx \leqq J(\mathfrak{c}, M_1, N_2),$$

which concludes the proof.

Lemma 9. *Let $\mathfrak{c}$ and $\mathfrak{d}$ be such that $|c_n| \leqq |d_n|$ $(n=1, 2, \cdots)$. Then for every N we have*

$$I(\mathfrak{c}, N) \leqq I(\mathfrak{d}, N).$$

The proof can be carried out exactly in the same way as that of Lemma V of TANDORI [3].

Lemma 10. *Let $\mathfrak{c}$ be such that $\|\mathfrak{c}\| < \infty$. Then there exists an increasing sequence $\{N_r\}_0^\infty$ of integers, $N_0 = 1$, with the following properties: for every* ONS $\{\varphi_n(x)\}$ *in* $[0, 1]$ *we have*

$$\sum_{r=1}^{\infty} \int_0^1 (t_{N_r}(x) - f(x))^2 \, dx < \infty, \text{[3]} \tag{13}$$

and, moreover,

$$\sum_{r=1}^{\infty} J(\mathfrak{c}, N_{r-1}, N_r) < \infty. \tag{14}$$

Proof. First we shall choose an increasing sequence $\{i_k\}_1^\infty$ of natural numbers for which

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (A_{i_k,n} - 1)^2 c^{n2} < \infty. \tag{15}$$

$\|\mathfrak{c}\| < \infty$ implies, using Lemma 1, $\mathfrak{c} \in l^2$. Thus there exist two sequences $\nu_1 < \nu_2 < \cdots$ and $i_1 < i_2 < \cdots$ of natural numbers such that

$$\sum_{n=\nu_k+1}^{\infty} c_n^2 \leqq \frac{1}{2^k},$$

and for every $n \leqq \nu_k$, making use of (2) and (3),

$$|A_{i_k,n} - 1| \leqq \left| \sum_{l=1}^{\infty} a_{i_k,l} - 1 \right| + \sum_{l=1}^{\nu_k} |a_{i_k,l}| \leqq \frac{1}{2^k} \qquad (k = 1, 2, \ldots).$$

[3]) $f(x)$ admits an expansion convergent in the mean: $f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$.

By (4) we get

$$\sum_{n=1}^{\infty}(A_{i_k,n}-1)^2 c_n^2 = \sum_{n=1}^{\nu_k}+\sum_{n=\nu_k+1}^{\infty} \leqq \frac{1}{2^{2k}}\sum_{n=1}^{\nu_k} c_n^2+4K^2\sum_{n=\nu_k+1}^{\infty} c_n^2 \leqq$$

$$\leqq \frac{1}{2^k}\left(\sum_{n=1}^{\infty} c_n^2+4K^2\right),$$

whence (15) follows.

For the sake of brevity we write $J(M, N)$ instead of $J(\mathfrak{c}, M, N)$ in the remaining part of the proof. Set $N_0=1$ and $N_1=i_1$. By Lemma 8 we can select an index $N_2=i_{k_1}$ with $k_1>1$ such that

$$J(1, N_1)+J(k, l)\leqq J(1, l)+\tfrac{1}{2},$$

whenever $N_2\leqq k<l$. In particular, replacing k by N_2 and l by $N_3=i_{k_1+1}$, we obtain

$$J(1, N_1)+J(N_2, N_3)\leqq J(1, N_3)+\tfrac{1}{2}.$$

Let us repeat the above argument. We get that there exists an index $N_4=i_{k_2}$ with $k_2>k_1+1$ such that

$$J(1, N_3)+J(k, l)\leqq J(1, l)+\tfrac{1}{4},$$

whenever $N_4\leqq k<l$, and in particular

$$J(1, N_3)+J(N_4, N_5)\leqq J(1, N_5)+\tfrac{1}{4}$$

with $N_5=i_{k_2+1}$. Continuing this procedure we obtain an infinite sequence $N_1<N_2<\cdots$ of indices such that we have

$$J(1, N_{2r-1})+J(k, l) \leqq J(1, l)+\frac{1}{2^r}, \tag{16}$$

whenever $l>k\geqq N_{2r}=i_{k_r}$ $(i_{k_r}>i_{k_{r-1}+1})$, and in particular

$$J(1, N_{2r-1})+J(N_{2r}, N_{2r+1}) \leqq J(1, N_{2r+1})+\frac{1}{2^r}, \tag{17}$$

where $N_{2r+1}=i_{k_r+1}$.

Let ϱ be a natural number. Let us consider the inequalities (17) in turn for $r=1, 2, \cdots, \varrho$, and add them. Then we get

$$\sum_{r=1}^{\varrho} J(1, N_{2r-1})+\sum_{r=1}^{\varrho} J(N_{2r}, N_{2r+1}) \leqq \sum_{r=1}^{\varrho} J(1, N_{2r+1})+1,$$

whence

$$\sum_{r=0}^{\varrho} J(N_{2r}, N_{2r+1}) \leqq J(1, N_{2\varrho+1})+1 \qquad (\varrho = 1, 2, \ldots). \tag{18}$$

By (16), putting $k=N_{2r+1}$ and $l=N_{2r+2}$, we obtain

$$J(1, N_{2r-1})+J(N_{2r+1}, N_{2r+2}) \leqq J(1, N_{2r+2})+\frac{1}{2^r} \qquad (r = 1, 2, \ldots). \tag{19}$$

Let us consider the inequality (19) for every $r=1, 2, \cdots, \varrho$. By adding them, and using the fact that $J(1, N)$ is a non-decreasing function of N, we get

$$\sum_{r=1}^{\varrho} J(1, N_{2r+2})+1 \geqq \sum_{r=1}^{\varrho} J(1, N_{2r-1})+\sum_{r=1}^{\varrho} J(N_{2r+1}, N_{2r+2}) \geqq$$

$$\geqq \sum_{r=1}^{\varrho} J(1, N_{2r-2})+\sum_{r=1}^{\varrho} J(N_{2r+1}, N_{2r+2}),$$

therefore, we have

$$\sum_{r=1}^{\varrho} J(N_{2r-1}, N_{2r}) \leqq J(1, N_{2\varrho})+J(1, N_{2\varrho+2})+1. \tag{20}$$

Combining the results (18) and (20), we obtain

$$\sum_{r=1}^{2\varrho+1} J(N_{r-1}, N_r) \leqq 3J(1, N_{2\varrho+2})+2 \leqq 12I(\mathfrak{c}, N_{2\varrho+2})+2.$$

As $\|\mathfrak{c}\|<\infty$, we get (14). Since $\{N_r\}$ a subsequence of $\{i_k\}$, (13) is also satisfied.

The proof of Lemma 10 is complete.

Lemma 11. *Let M and N be natural numbers, $M<N$. There exists an* ONS $\{\psi_n(x)\}_1^\infty$ *of step functions in* $[0, 1]$ *and an interval* $E \subseteqq [0, \frac{1}{2}]$ *having the following properties:*

$$\max_{M \leqq i<j \leqq N} \left| \sum_{n=1}^{\infty} (A_{jn}-A_{in}) c_n \psi_n(x) \right| \geqq 2 \quad \textit{if} \quad x \in E,$$

and

$$|E| \geqq K_1 \min\left(\frac{1}{2}, J(\mathfrak{c}, M, N)\right).^{4)}$$

Proof. According to the definition of J there exists an ONS $\{\varphi_n(x)\}_1^\infty$ in $[0, 1]$ such that

$$\int_0^1 \left(\max_{M \leqq i<j \leqq N} \left| \sum_{n=1}^{\infty} (A_{jn}-A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \geqq \frac{1}{2} J(\mathfrak{c}, M, N).$$

By virtue of Lemma 3 there exists a natural number ν_0 such that

$$\int_0^1 \left(\max_{M \leqq i<j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn}-A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \geqq \frac{1}{4} J(\mathfrak{c}, M, N). \tag{21}$$

Let ε be an arbitrary positive real number, $\varepsilon<1$. We consider a system $\{\chi_n(x)\}_1^{\nu_0}$ of step functions for which

$$\int_0^1 (\varphi_n(x)-\chi_n(x))^2 dx \leqq \varepsilon^2 \qquad (n=1,2, \ldots, \nu_0).$$

4) $|E|$ denotes the Lebesgue measure of the set E.

Set

$$\alpha_{ln} = \int_0^1 \chi_l(x)\chi_n(x)\,dx \qquad (l, n = 1, 2, \dots, \nu_0),$$

and

$$\eta_n = \sum_{l=1}^{n-1} |\alpha_{ln}| + \sum_{l=n+1}^{\nu_0} |\alpha_{ln}| \qquad (n = 1, 2, \dots, \nu_0).$$

We get by a simple calculation that

$$\int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \chi_n(x) \right| \right)^2 dx \geqq \frac{1}{8} J(\mathfrak{c}, M, N), \tag{22}$$

and

$$\int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \left(1 - \frac{1}{\sqrt{\alpha_{nn} + \eta_n}}\right) \chi_n(x) \right| \right)^2 dx \leqq \frac{1}{16} J(\mathfrak{c}, M, N), \tag{23}$$

provided ε is small enough [5]).

Now we continue $\chi_n(x)$ on $[0, 2]$ so that we divide $(1, 2]$ into as many equal parts as there exist pairs of numbers l, n with $1 \leqq l, n \leqq \nu_0, l \neq n$. We denote the single subintervals by I_{ln}, and then define for $x \in (1, 2]$ the values of the function $\chi_n(x)$ $(n \leqq \nu_0)$ as follows:

$$\chi_n(x) = \begin{cases} \sqrt{\frac{1}{2}\nu_0(\nu_0 - 1)\,|\alpha_{ln}|} & x \in I_{nl}, \\ -\sqrt{\frac{1}{2}\nu_0(\nu_0 - 1)\,|\alpha_{ln}|}\ \operatorname{sign} \alpha_{ln} & x \in I_{ln} \quad (l = 1, 2, \dots, \nu_0;\ l \neq n). \end{cases}$$

The functions $\chi_n(x)$ are orthogonal to each other in $[0, 2]$ since for $l \neq n$

$$\int_0^2 \chi_l(x)\chi_n(x)\,dx = \int_0^1 + \int_1^2 = \int_0^1 + \int_{I_{ln}} + \int_{I_{nl}} = \alpha_{ln} - |\alpha_{ln}| \operatorname{sign} \alpha_{ln} = 0.$$

Furthermore, we have

$$\int_0^2 \chi_n^2(x)\,dx = \int_0^1 \chi_n^2(x)\,dx + \sum_{l=1}^{n-1} |\alpha_{ln}| + \sum_{l=n+1}^{\nu_0} |\alpha_{ln}| = \alpha_{nn} + \eta_n.$$

Setting

$$\bar{\chi}_n(x) = \frac{1}{\sqrt{\alpha_{nn} + \eta_n}} \chi_n(x),$$

we get an ONS of step functions in $[0, 2]$, and from (22) and (23) we obtain

$$\int_0^2 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \bar{\chi}_n(x) \right| \right)^2 dx \geqq \frac{1}{32} J(\mathfrak{c}, M, N). \tag{24}$$

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[5]) To show (22), we can, for example, use the inequality $(a+b)^2 \geqq a^2 - 2|a||b|$, and to show (23), we make use of another inequality $\left|1 - 1/\sqrt{1+a}\right| \leqq |a|$ if $a > K_2$, where $-1 < K_2 < 0$.

Let us consider the step function

$$S(x) = \max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \bar{\chi}_n(x) \right|.$$

We can divide [0, 2] into a finite number of subintervals $J_1, J_2, \cdots, J_r$ such that $S(x)$ has a constant value w_ϱ on each subinterval J_ϱ $(\varrho = 1, 2, \cdots, r)$. Set

$$\sum_{\varrho=1}^{r} w_\varrho^2 |J_\varrho| = A.$$

Without loss of generality, we may assume that $A \leqq 2$. Putting

$$u_0 = 0, \quad u_\varrho = \frac{1}{4} \sum_{\sigma=1}^{\varrho} w_\sigma^2 |J_\sigma| \qquad (\varrho = 1, 2, \dots, r),$$

and

$$\bar{\varphi}_n(x) = \begin{cases} \dfrac{2}{w_{\varrho+1}} \bar{\chi}_n \left(\dfrac{4}{w_{\varrho+1}^2} (x - u_\varrho) + \sum_{\sigma=1}^{\varrho} |J_\sigma| \right) & \text{if } \ x \in [u_\varrho, u_{\varrho+1}) \\ & (w_\varrho \neq 0;\ \varrho = 0, 1, \dots, r-1), \\ 0 \quad \text{otherwise in} \quad [0, 1], \end{cases}$$

we can see that $\{\bar{\varphi}_n(x)\}_1^{\nu_0}$ is an ONS in [0, 1]. Set $E = [0, u_r)$. It is clear that $E \subseteqq [0, \frac{1}{2}]$, and by virtue of (24)

$$|E| \geqq \min \left(\frac{1}{2}, \frac{1}{32} J(\mathfrak{c}, M, N) \right).$$

On account of the definition of the functions $\bar{\varphi}_n(x)$, we have

$$\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \bar{\varphi}_n(x) \right| \geqq 2 \quad \text{if} \quad x \in E. \tag{25}$$

Since the functions $\bar{\varphi}_n(x)$ with $n \leqq \nu_0$ identically vanish outside $[0, \frac{1}{2}]$, we can give an ONS $\{\psi_n(x)\}_1^\infty$ of step functions in [0, 1] in a trivial manner such that we have $\psi_n(x) = \bar{\varphi}_n(x)$ for $n \leqq \nu_0$, and $\psi_n(x) = 0$ if $x \in [0, \frac{1}{2}]$ for every $n \geqq \nu_0 + 1$. This does not affect the inequality (25), and concludes the proof of Lemma 11.

§2. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. (A) *Sufficiency.* Assume that $\|\mathfrak{c}\| < \infty$. By virtue of Lemma 10 there exists an increasing sequence $\{N_r\}$ of natural numbers such that both (13) and (14) are convergent. Applying B. Levi's theorem, we get on the one hand that the subsequence $\{t_{N_r}(x)\}$ converges a.e., on the other hand that

$$\delta_r(x) = \max_{N_r \leqq i < j \leqq N_{r+1}} |t_j(x) - t_i(x)| \to 0 \qquad (r \to \infty).$$

It is obvious that for $N_{r-1} < n < N_r$

$$|t_n(x) - t_{N_r}(x)| \leqq \delta_r(x) \to 0 \qquad (r \to \infty),$$

and the proof of the sufficiency is complete.

In the course of this proof we have obtained the following result: *if there exists an increasing sequence $\{N_r\}$ of integers such that both the subsequence $\{t_{N_r}(x)\}$ is convergent a.e. and*

$$\sum_{r=1}^{\infty} J(\mathfrak{c}, N_{r-1}, N_r) < \infty$$

holds, then the series (1) *is T-summable a.e.*

(B) *Necessity.* Suppose $\|\mathfrak{c}\| = \infty$. Using Lemma 6, we get that for any fixed natural number M

(26) $$\lim_{N \to \infty} J(\mathfrak{c}, M, N) = \infty$$

holds. We shall define by induction two sequences $1 = M_1 < N_1 < M_2 < N_2 < \cdots$ and $0 = \nu_1 < \nu_2 < \cdots$ of integers, depending only on T and $\mathfrak{c}$, such that

(27) $$J(\mathfrak{c}, M_r, N_r) \geqq 1 \qquad (r = 1, 2, \ldots),$$

(28) $$\sum_{r=1}^{\infty} J(P^{\nu_r}\mathfrak{c}, M_r, N_r) < \infty,$$

and

(29) $$\sum_{r=1}^{\infty} J(P_{\nu_{r+1}+1}\mathfrak{c}, M_r, N_r) < \infty$$

hold.

First let $r = 1$. By virtue of (26) there exists a natural number N_1 for which

$$J(\mathfrak{c}, 1, N_1) \geqq 1.$$

Applying Lemma 3, there exists another natural number ν_2 such that

$$J(P_{\nu_2+1}\mathfrak{c}, 1, N_1) \leqq \frac{1}{2}.$$

Now $r \geqq 1$ being arbitrary, we assume that $M_\varrho, N_\varrho, \nu_{\varrho+1}$ with $\varrho = 1, 2, \cdots, r-1$ are already defined. According to Lemma 4 there exists a natural number $M_r > N_{r-1}$ such that for every $N > M_r$ we have

(30) $$J(P^{\nu_r}\mathfrak{c}, M_r, N) \leqq \frac{1}{2^r}.$$

By (26) we can choose a natural number $N_r > M_r$ such that

$$J(\mathfrak{c}, M_r, N_r) \geqq 1.$$

(30) holds if N is replaced by N_r in it. Finally using Lemma 3, we obtain a natural number ν_{r+1} for which

$$J(P_{\nu_{r+1}+1}\mathfrak{c}, M_r, N_r) \leqq \frac{1}{2^r}.$$

Thus M_r, N_r and ν_{r+1} will be defined by induction for every $r \geqq 1$ in such a manner that the relations (27), (28) and (29) will be satisfied.

After these preliminaries, we begin with applying Lemma 11 by choosing subsequently M_r and N_r $(r=1, 2, \cdots)$ (instead of M and N). Denote by $\{\psi_n^{(r)}(x)\}_1^\infty$ the corresponding ONS of step functions in $[0, 1]$ and by E_r $(r=1, 2, \cdots)$ the corresponding intervals in the sense of Lemma 11. That is, for every $r \geqq 1$ we have the following properties:

$$\max_{M_r \leqq i < j \leqq N_r} |t_j^{(r)}(x) - t_i^{(r)}(x)| \geqq 2 \tag{31}$$

in the points of the interval $E_r \subseteqq [0, \frac{1}{2}]$ with

$$|E_r| \geqq K_1 \min\left(\frac{1}{2}, J(\mathfrak{c}, M_r, N_r)\right) = \frac{K_1}{2}, \tag{32}$$

where $t_i^{(r)}(x)$ denotes the ith T-mean of the series $\Sigma c_n \psi_n^{(r)}(x)$.

We are going to define a system $\{\Phi_n(x)\}_1^\infty$ of orthonormal step functions in $[0, 1]$, and a stochastically independent sequence $\{F_r\}_1^\infty$ of simple sets [6]) having the following properties: for every $x \in F_r$ there exists a point $y \in E_r$ for which

$$\max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_r+1}^{\nu_{r+1}} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| = \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_r+1}^{\nu_{r+1}} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right|, \tag{33}$$

and

$$|F_r| = |E_r| \qquad (r = 1, 2, \ldots). \tag{34}$$

The construction will be accomplished by recurrence with respect to r. First, let $r=1$. Writing

$$\Phi_n(x) = \psi_n^{(1)}(x) \qquad (n = 1, 2, \ldots, \nu_2),$$

and

$$F_1 = E_1,$$

we can see that (33) and (34) are satisfied.

Now we suppose that all the orthonormal step functions $\Phi_n(x)$ with $n=1, 2, \cdots, \nu_r$ and the stochastically independent simple sets F_ϱ with $\varrho=1, 2, \cdots, r-1$ are already determined and satisfy (33) and (34). Then we can divide $[0, 1]$ into a finite number of subintervals $I_1, I_2, \cdots, I_Q$, in which every function $\Phi_n(x)$ $(n \leqq \nu_r)$ remains constant and every simple set F_ϱ $(\varrho \leqq r-1)$ is the union

[6]) A set F is called simple if it is the union of finitely many non-overlapping intervals.

of a finite number of I_q $(1 \leqq q \leqq Q)$. Let I'_q, I''_q denote the two halves of the interval I_q. Now let us put for $\nu_r < n \leqq \nu_{r+1}$

$$\Phi_n(x) = \sum_{q=1}^{Q} \psi_n^{(r)}(I'_q; x) - \sum_{q=1}^{Q} \psi_n^{(r)}(I''_q; x),$$

and

$$F_r = \bigcup_{q=1}^{Q} (E_r(I'_q) \cup E_r(I''_q)),$$

where $f(I; x)$ denotes the function arising from $f(x)$ as the result of the linear transformation of the interval $[0, 1]$ into its subinterval $I = [u, v]$, i.e.

$$f(I; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{if} \quad x \in (u, v), \\ 0 & \text{otherwise;} \end{cases}$$

furthermore, let $E(I)$ denote the image set of E arising from this transformation. It is obvious that the step functions $\Phi_n(x)$ with $n = 1, 2, \cdots, \nu_{r+1}$ are orthonormal, the simple sets F_ϱ with $\varrho = 1, 2, \cdots, r$ are stochastically independent, (33) holds for r, and

$$|F_r| = \sum_{q=1}^{Q} (|E_r(I'_q)| + |E_r(I''_q)|) = |E_r| \sum_{q=1}^{Q} (|I'_q| + |I''_q|) = |E_r|,$$

i.e. (34) is also satisfied for r. Thus $\{\Phi_n(x)\}_1^\infty$ and $\{F_r\}_1^\infty$ will be given by induction.

To finish the proof of the necessity, we have to show that the series

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) \tag{35}$$

fails at almost every point x to be T-summable. For the sake of simplicity, let us denote the ith T-mean of (35) by $T_i(x)$. Taking into account (33), let us consider the following inequality for every $r \geqq 1$

$$\max_{M_r \leqq i < j \leqq N_r} |T_j(x) - T_i(x)| \geqq \max_{M_r \leqq i < j \leqq N_r} |t_j^{(r)}(y) - t_i^{(r)}(y)| -$$

$$- \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=1}^{\nu_r} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| - \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_{r+1}+1}^{\infty} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| -$$

$$- \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=1}^{\nu_r} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right| - \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_{r+1}+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right|,$$

where $x \in F_r$ and $y \in E_r$ in the sense of (33). We show that the last four maxima on the right-hand side of this inequality tend to 0 as $r \to \infty$. In fact, this follows by virtue of (28) and (29), using B. LEVI's theorem. More precisely, there exists a set

G of measure zero such that for every $x\in[0, 1]-G$ we have

$$\overline{\lim_{r\to\infty}}\Big(\max_{M_r\leqq i<j\leqq N_r}|T_j(x)-T_i(x)|\Big)\geqq \overline{\lim_{r\to\infty}}\Big(\max_{M_r\leqq i<j\leqq N_r}|t_j^{(r)}(y)-t_i^{(r)}(y)|\Big).$$

Since the sets F_r are stochastically independent, by (32) and (34), we get

$$\Big|\overline{\lim_{r\to\infty}} F_r\Big| = 1.$$

Thus, on account of (31), we obtain that

$$\overline{\lim_{r\to\infty}}\Big(\max_{M_r\leqq i<j\leqq N_r}|T_j(x)-T_i(x)|\Big)\geqq 2$$

holds whenever

$$x\in\overline{\lim_{r\to\infty}} F_r - G,$$

that is, for almost every $x\in[0, 1]$.

The proof of the necessity is now complete.

To accomplish the proof of Theorem 1, we have to show that the assertions concerning $\|\mathfrak{c}\|$ are also fulfilled. Let us define the usual vector operations in B as follows:

$$\mathfrak{c}+\mathfrak{d} = \{c_n+d_n\}_1^\infty, \quad \alpha\mathfrak{c} = \{\alpha c_n\}_1^\infty.$$

It is obvious that B is a linear space. From Lemma 1 we infer

$$\left\{\sum_{n=1}^\infty c_n^2\right\}^{1/2} \leqq \|\mathfrak{c}\| \leqq K\sum_{n=1}^\infty |c_n|. \tag{36}$$

$\|\mathfrak{c}\|$ is a norm in B, for (i) $\|\mathfrak{c}\|=0$ if and only if $c_n=0$ for every n; (ii) $\|\alpha\mathfrak{c}\|=|\alpha|\cdot\|\mathfrak{c}\|$ for every real number α; (iii) $\|\mathfrak{c}+\mathfrak{d}\|\leqq\|\mathfrak{c}\|+\|\mathfrak{d}\|$. (i) follows from (36), (ii) is obvious, (iii) follows from Lemma 5.

We prove that B is a complete space. For this purpose, let $\mathfrak{c}_m=\{c_{mn}\}_{n=1}^\infty\in B$ $(m=1, 2, \cdots)$ be for which

$$\|\mathfrak{c}_{m'}-\mathfrak{c}_{m''}\|\to 0 \qquad (m', m''\to\infty).$$

By virtue of (36), we get for every n

$$c_{mn}\to c_n \qquad (m\to\infty).$$

Let ε be an arbitrary positive real number. According to the definition of the norm, we have

$$I(\mathfrak{c}_{m'}-\mathfrak{c}_{m''}, N)\leqq\varepsilon^2 \qquad (m', m''\geqq\mu(\varepsilon))$$

for every N, and, by virtue of Lemma 2, for every ν

$$I\big(P^\nu(\mathfrak{c}_{m'}-\mathfrak{c}_{m''}), N\big)\leqq\varepsilon^2 \qquad (m', m''\geqq\mu(\varepsilon)).$$

For m' fixed and m'' tending to infinity, by Lemma 7, we get

$$I\big(P^\nu(\mathfrak{c}_{m'}-\mathfrak{c}), N\big)\leqq\varepsilon^2 \qquad (m'\geqq\mu(\varepsilon))$$

for every ν and N. Hence, applying Lemma 3, we obtain

$$I(\mathfrak{c}_{m'}-\mathfrak{c}, N) \leqq \varepsilon^2 \qquad (m' \geqq \mu(\varepsilon))$$

for every N, where $\mathfrak{c}=\{c_n\}_1^\infty$, and consequently

$$\|\mathfrak{c}_{m'}-\mathfrak{c}\| \leqq \varepsilon \qquad (m' \geqq \mu(\varepsilon)).$$

So we have, by (iii), $\mathfrak{c}\in B$ and, moreover,

$$\|\mathfrak{c}_m-\mathfrak{c}\| \to 0 \qquad (m\to\infty),$$

which was to be proved.

This concludes the proof of Theorem 1.

Proof of Theorem 2. If (1) is T-summable a.e. for every ONS in [0, 1], then by virtue of Theorem 1, we have $\|\mathfrak{c}\|<\infty$. Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ by [0, 1], and denote by $t_i(x)$ the ith T-mean of (1). From Lemma 10, applying B. Levi's theorem, we get that the series

$$\sum_{r=1}^{\infty}\left(t_{N_r}(x)-f(x)\right)^2$$

converges a.e. Let us denote by $F(x)$ the positive square root of the sum of this er It is obvious that $F(x)$ is a square integrable function, the square integral of which depends only on the coefficients c_n. By (14), it follows that the function

$$G(x)=\left\{\sum_{r=1}^{\infty}\left(\max_{N_{r-1}\leqq i<j\leqq N_r}|t_j(x)-t_i(x)|\right)^2\right\}^{1/2}$$

is square integrable; its square integral depends only on the coefficients c_n. Let be an arbitrary index with $N_{r-1}<i\leqq N_r$. It is clear that

$$|t_i(x)| \leqq |t_i(x)-t_{N_r}(x)|+|t_{N_r}(x)-f(x)|+|f(x)| \leqq G(x)+F(x)+|f(x)|.$$

This completes the proof.

§3. Proofs of Theorems 3—5

Theorem 3 follows immediately from Lemma 9.

Proof of Theorem 4. Let ε be a positive real number, given in advance, furthermore, let $\{\varphi_n(x)\}$ be an arbitrary ONS in [0, 1]. We denote by $s_k^{(m)}(x)$ the kth partial sum of the series

$$\sum_{n=1}^{\infty} c_{mn}\varphi_n(x)$$

and by $t_i^{(m)}(x)$ the ith T-mean. By Theorem 3, $\|\mathfrak{c}_1\|<\infty$ implies $\|\mathfrak{c}_m\|<\infty$ and so

$\mathfrak{c}_m \in l^2$ for every m. By the Riesz—Fischer theorem there exists a square integrable function $f_m(x)$ such that $\{s_k^{(m)}(x)\}$ converges in the mean to $f_m(x)$ as $k\to\infty$, and so does $\{t_i^{(m)}(x)\}$ as $i\to\infty$.

Since $\|\mathfrak{c}_1\|<\infty$, by virtue of Lemma 10 there exists an increasing sequence $\{N_r\}$ of integers such that $N_0=1$,

$$\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}(A_{N_r,n}-1)^2 c_{1n}^2<\infty, \tag{37}$$

and

$$\sum_{r=1}^{\infty} J(\mathfrak{c}_1, N_{r-1}, N_r)<\infty. \tag{38}$$

Let us consider the following inequalities:

$$\max_{1\le i\le N}|t_i^{(m)}(x)| \le |f_m(x)|+\left\{\sum_{r=1}^{\infty}\left(f_m(x)-t_{N_r}^{(m)}(x)\right)^2\right\}^{1/2}+$$
$$+\left\{\sum_{r=1}^{\infty}\left(\max_{N_{r-1}\le i<j\le N_r}|t_j^{(m)}(x)-t_i^{(m)}(x)|\right)^2\right\}^{1/2},$$

whence

$$\int_0^1\left(\max_{1\le i\le N}|t_i^{(m)}(x)|\right)^2dx\le 3\left(\sum_{n=1}^{\infty}c_{mn}^2+\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}(A_{N_r,n}-1)^2c_{mn}^2+\sum_{r=1}^{\infty}J(\mathfrak{c}_m, N_{r-1}, N_r)\right)$$

for every $m\ge 1$ and $N\ge 1$. By (37) and (38), we can choose the natural numbers ϱ_0 and ν_0 so that

$$\sum_{n=\varrho_0+1}^{\infty}c_{1n}^2\le\varepsilon^2,\qquad \sum_{r=\varrho_0+1}^{\infty}\sum_{n=1}^{\infty}(A_{N_r,n}-1)^2c_{1n}^2\le\varepsilon^2,$$

$$\sum_{r=\varrho_0+1}^{\infty}J(\mathfrak{c}_1, N_{r-1}, N_r)\le\varepsilon^2,\qquad \sum_{r=1}^{\varrho_0}\sum_{n=\nu_0+1}^{\infty}(A_{N_r,n}-1)^2c_{1n}^2\le\varepsilon^2$$

are satisfied. The coefficients c_{mn} being decreasing in m for every fixed n, we obtain

$$\int_0^1\left(\max_{1\le i\le N}|t_i^{(m)}(x)|\right)^2dx\le$$

$$\le 3\left(\sum_{n=1}^{\varrho_0}c_{mn}^2+\sum_{r=1}^{\varrho_0}\sum_{n=1}^{\nu_0}(A_{N_r,n}-1)^2c_{mn}^2+\sum_{r=1}^{\varrho_0}J(\mathfrak{c}_m, N_{r-1}, N_r)\right)+12\varepsilon^2. \tag{39}$$

By a simple calculation we get

$$\sum_{r=1}^{\varrho_0}J(\mathfrak{c}_m, N_{r-1}, N_r)\le 2\sum_{r=1}^{\varrho_0}J(P^\lambda\mathfrak{c}_m, N_{r-1}, N_r)+2\sum_{r=1}^{\varrho_0}J(P_{\lambda+1}\mathfrak{c}_m, N_{r-1}, N_r), \tag{40}$$

where the natural number λ is fixed in such a manner that

$$(41)\quad \sum_{r=1}^{\varrho_0} J(P_{\lambda+1}\mathfrak{c}_m, N_{r-1}, N_r) \leqq 4 \sum_{r=1}^{\varrho_0} I(P_{\lambda+1}\mathfrak{c}_m, N_r) \leqq 4\varrho_0 I(P_{\lambda+1}\mathfrak{c}_1, N_{\varrho_0}) \leqq \varepsilon^2.$$

Here we took Lemma 9 and Lemma 3 into consideration.

By (39), (40) and (41), on account of Lemma 7, we get that there exists a natural number $\mu(\varepsilon)$ such that

$$\int_0^1 \Big(\max_{1\leqq i\leqq N} |t_i^{(m)}(x)|\Big)^2 dx \leqq 16\varepsilon^2 \qquad (m\geqq \mu(\varepsilon)).$$

Since $\{\varphi_n(x)\}$ is an arbitrary ONS, thus we obtain for every N

$$I(\mathfrak{c}_m, N) \leqq 16\varepsilon^2 \qquad (m\geqq\mu(\varepsilon)),$$

and consequently

$$\|\mathfrak{c}_m\| \leqq 4\varepsilon \qquad (m\geqq\mu(\varepsilon)),$$

which is what had to be proved.

Proof of Theorem 5. If $\mathfrak{c}\in B$ then, according to Theorem 4, we have

$$\|P^\nu\mathfrak{c}-\mathfrak{c}\|\to 0 \qquad (\nu\to\infty).$$

Hence the class of all the finite sequences is everywhere dense in B. Applying the continuity we infer that every finite sequence can be approximated, as closely as we wish, by a finite sequence of rational numbers. But all the finite sequences of rational numbers form a countable set. So we have proved that B is separable.

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