A note on the strong T-summation of orthogonal series

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1. Let $\{\varphi_k(x)\}$ $(k=0, 1, \cdots)$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{1}$$

with real coefficients satisfying

$$\sum_{k=0}^{\infty} a_k^2 < \infty. \tag{2}$$

By the Riesz—Fischer theorem, the series (1) converges in the mean to a square-integrable function $f(x)$. We denote the kth partial sum of the series (1) by $s_k(x)$.

Let $T=(\alpha_{ik})$ $(i, k=0, 1, \cdots)$ be a double infinite matrix of real numbers. We say that the series (1) is *T-summable* to $f(x)$ at the point $x \in (a, b)$ if

$$t_i = \sum_{k=0}^{\infty} \alpha_{ik} s_k(x)$$

exists for all i (except perhaps finitely many of them), and

$$\lim_{i \to \infty} t_i(x) = f(x).$$

The series (1) is called *strongly T-summable* at the point x if the relation

$$\lim_{i \to \infty} \sum_{k=0}^{\infty} \alpha_{ik} (s_k(x) - f(x))^2 = 0$$

holds.

A T-summation process is called *permanent* if $\lim_{k \to \infty} s_k = s$ always implies $\lim_{i \to \infty} t_i = s$. Necessary and sufficient conditions for the permanence of a summation process are well known. (See ALEXITS [1], p. 65.)

2. In the most frequently used cases T-summability and strong T-summability of the series (1) coincide under the condition (2), up to sets of measure zero. For the classical $(C, 1)$-summation process this was proved by ZYGMUND [9] (see also

TANDORI [8]), for $(C, \beta>0)$-summation by SUNOUCHI [7], and for Riesz summation by MEDER [4] and LEINDLER [2]. (In the latter case

$$\alpha_{ik} = \frac{\lambda_{k+1}-\lambda_k}{\lambda_{i+1}} \quad \text{for} \quad k \leqq i, \quad \alpha_{ik}=0 \quad \text{for} \quad k>i,$$

where $\{\lambda_i\}$ is a strictly increasing sequence of positive real numbers with $\lambda_0=0$ and $\lambda_i \to \infty$.) Finally, for the de la Vallée Poussin summation, this was proved also by LEINDLER [3]. (In this case

$$\alpha_{ik} = \frac{1}{\mu_i} \quad \text{if} \quad k = i-\mu_i+1, \quad i-\mu_i+2, \ldots, i;$$

$$\alpha_{ik} = 0 \quad \text{if} \quad k = 0, 1, \ldots, i-\mu_i; \quad i+1, \quad i+2, \ldots,$$

where $\{\mu_i\}$ is an increasing sequence of natural numbers with $\mu_{i+1}-\mu_i \leqq 1$.)

3. These particular results raise the following question: does, under condition (2), T-summability of the series (1) almost everywhere imply strong T-summability for any T-process?

In this paper we show that the answer is in general negative. We prove the following

Theorem. *There exist a uniformly bounded orthonormal system $\{\Phi_k(x)\}$ in $(0, 1)$, a sequence $\{c_k\}$ of coefficients and a permanent T-summation process such that*

$$\sum_{k=0}^{\infty} c_k^2 < \infty$$

is satisfied, the orthonormal series

$$\sum_{k=0}^{\infty} c_k \Phi_k(x) \tag{3}$$

is T-summable almost everywhere, but the relation

$$\overline{\lim_{i\to\infty}} \sum_{k=0}^{\infty} \alpha_{ik} |s_k(x)-f(x)|^{\gamma} = \infty \tag{4}$$

holds almost everywhere in $(0, 1)$ for any $\gamma>0$.

The proof will be accomplished by direct construction, the T-summation in question being defined by a method due to MENCHOFF [6].

4. We require some lemmas. In the sequel, we use $C, C_1, C_2, \cdots$ to denote positive constants.

Lemma 1. (MENCHOFF [5]) *Let $\nu > 3$ be a natural number and let $C > 1$. Then there exists in $(-1, C)$ a system $\{\psi_{k\nu}(x)\}$ $(1 \leqq k \leqq \nu^2)$ of orthonormal step functions with the following properties:*

(i) $|\psi_{k\nu}(x)| \leqq C_1$ $(1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C)$;

(ii) *for every point* $x \in (\frac{1}{2}, 1)$ *there exists an index* $l(x)$ *depending on* x $\left(1 \leqq l(x) \leqq \nu^2\right)$ *such that*

$$\sum_{k=1}^{l(x)} \psi_{k\nu}(x) \geqq C_2 \nu \log \nu.$$

Let us define another system $\{\chi_{k\nu}(x)\}$ $(1 \leqq k \leqq 2\nu^2)$ of orthonormal step functions in $(-2, C)$ as follows:

$$\chi_{k\nu}(x) = \chi_{\nu^2+k,\nu}(x) = \frac{1}{\sqrt{2}} \psi_{k\nu}(x) \qquad (1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C), \tag{5}$$

$$\chi_{k\nu}(x) = \frac{1}{\sqrt{2}} r_k(x+2), \quad \chi_{\nu^2+k,\nu}(x) = -\frac{1}{\sqrt{2}} r_k(x+2) \qquad (1 \leqq k \leqq \nu^2,\ -2 \leqq x < -1),$$

where $r_k(x) = \operatorname{sign} \sin 2^k \pi x$ denotes the kth Rademacher function $(k = 0, 1, \cdots)$. By virtue of Lemma 1 it is clear that

$$|\chi_{k\nu}(x)| \leqq C_3 \qquad (1 \leqq k \leqq 2\nu^2,\ -2 \leqq x \leqq C);$$

furthermore, for every point $x \in (\frac{1}{2}, 1)$ there exists an index $l(x)$ $\left(1 \leqq l(x) \leqq \nu^2\right)$ such that

$$\sum_{k=1}^{l(x)} \chi_{k\nu}(x) = \sum_{k=\nu^2+1}^{\nu^2+l(x)} \chi_{k\nu}(x) \geqq C_4 \nu \log \nu. \tag{6}$$

This construction can also be found in the cited paper of MENCHOFF [6].

5. Proof of the theorem. Let $g(y)$ be an arbitrary function defined in $(-2, C)$ and let $I = (u, v)$ be an arbitrary finite interval. We proceed from the interval I to the interval $(-2, C)$ by means of the linear transformation

$$y = -2 + \frac{x-u}{v-u}(2+C) \qquad (u \leqq x \leqq v,\ -2 \leqq y \leqq C),$$

and put

$$g(I; x) = \begin{cases} \sqrt{2+C}\, g(y) & \text{if } u \leqq x \leqq v, \\ 0 & \text{elsewhere.} \end{cases}$$

Further, let $E(I)$ denote the image set of an arbitrary set $E \subset (-2, C)$ arising from this transformation. It is obvious that

$$\int_u^v g^2(I; x)\,dx = |I| \int_{-2}^{C} g^2(y)\,dy.\ ^{1)}$$

[1]) $|I|$ denotes the Lebesgue measure of the set I.

We are going to construct the system $\{\Phi_k(x)\}$ and an auxiliary system $\{\Psi_k(x)\}$ which has an important role in the proof. Let $\{\nu_r\}$ be any sequence of natural numbers, with $\nu_r > 3$ $(r=1, 2, \cdots)$, and let

$$N_0 = 0, \quad N_r = 2\sum_{\varrho=1}^{r} \nu_\varrho^2, \quad N_r' = N_{r-1} + \nu_r^2 \qquad (r=1, 2, \ldots).$$

First we set

$$\Phi_k(x) = \Psi_k(x) = r_k(x) \qquad (k = 0, 1, \ldots, N_1; 0 \leqq x \leqq 1).$$

Now $r>1$ being arbitrary, we assume that the step functions $\Phi_k(x)$, $\Psi_k(x)$ $(k=0, 1, \cdots, N_{r-1})$ are already defined. Then we divide $(0, 1)$ into a finite number of mutually disjoint subintervals $I_1, I_2, \cdots, I_s$, in which every function $\Phi_k(x)$, $\Psi_k(x)$ with $k \leqq N_{r-1}$ is constant. Let I_σ', I_σ'' denote the two halves of the interval I_σ, and set

$$\Phi_k(x) = \begin{cases} \chi_{k-N_{r-1}, \nu_r}(I_\sigma'; x) & \text{if } x \in I_\sigma' \\ -\chi_{k-N_{r-1}, \nu_r}(I_\sigma''; x) & \text{if } x \in I_\sigma'' \end{cases} \quad (\sigma = 1, 2, \ldots, s;\ N_{r-1} < k \leqq N_r);$$

$$\Psi_k(x) = \begin{cases} \dfrac{1}{\sqrt{2+C}}\, r_{k-N_{r-1}}(\tilde{I}_\sigma; x) & \text{if } x \in \tilde{I}_\sigma \text{ and } N_{r-1} < k \leqq N_r', \\[2ex] -\dfrac{1}{\sqrt{2+C}}\, r_{k-N_r'}(\tilde{I}_\sigma; x) & \text{if } x \in \tilde{I}_\sigma \text{ and } N_r' < k \leqq N_r, \end{cases}$$

where $\tilde{I}_\sigma$ can be either I_σ' or I_σ'' $(\sigma = 1, 2, \cdots, s)$. It is clear that these functions are also step functions.

Set $E_1 = (-2, -1)$, $E_2 = (-1, C)$ and $E_3 = (\frac{1}{2}, 1)$; furthermore, write

$$G_r'(1) = \bigcup_{\sigma=1}^{s} E_1(I_\sigma'), \quad G_r''(1) = \bigcup_{\sigma=1}^{s} E_1(I_\sigma''),$$

and

$$G_r(i) = \bigcup_{\sigma=1}^{s} \left(E_i(I_\sigma') \cup E_i(I_\sigma'')\right) \qquad (i=2, 3).$$

It is obvious that the interval $(0, 1)$ is the union of the mutually disjoint subsets $G_r'(1)$, $G_r''(1)$ and $G_r(2)$, and that

$$|G_r(3)| = \frac{1}{2(2+C)} \qquad (r=1, 2, \ldots). \tag{7}$$

We can easily prove that the system $\{\Phi_k(x)\}$ as constructed from the previously defined functions is orthonormal and uniformly bounded. Furthermore, the system $\{\Psi_k(x)\}$ can be divided into two subsystems, both of which are orthonormal. More exactly, MENCHOFF [6] proved the following

Lemma 2. *Let* $\{\Psi_k(x)\}$ *be the system of functions in* $(0, 1)$ *defined above, and set*

$$S' = \bigcup_{r=1}^{\infty} \{\Psi_k(x): N_{r-1} < k \leqq N'_r\}, \qquad S'' = \bigcup_{r=1}^{\infty} \{\Psi_k(x): N'_r < k \leqq N_r\}.$$

Then both S' *and* S'' *are orthonormal convergence systems*[2]).

6. We define the matrix $T=(\alpha_{ik})$ $(i, k=0, 1, \cdots)$ as follows

$$\alpha_{00}=1 \quad \text{and} \quad \alpha_{0k}=0 \quad \text{for} \quad k \geqq 1,$$

and in general, for an arbitrary natural number $r(\geqq 1)$, we distinguish two subcases: if $N_{r-1} < i \leqq N'_r$, then we set

$$\alpha_{ii} = \alpha_{i,\, v_r^2+i} = \frac{1}{2} \quad \text{and} \quad \alpha_{ik}=0 \quad \text{otherwise};$$

if $N'_r < i \leqq N_r$, then

$$\alpha_{i,N_r}=1 \quad \text{and} \quad \alpha_{ik}=0 \quad \text{otherwise}.$$

From the definition of the matrix T we can immediately infer the permanence of the T-summation process.

7. We define the sequence $\{c_k\}$ $(c_0=0)$ of coefficients as follows

$$c_k = \begin{cases} p_r & \text{if} \quad N_{r-1} < k \leqq N'_r, \\ -p_r & \text{if} \quad N'_r < k \leqq N_r \end{cases} \qquad (r=1, 2, \ldots),$$

where the sequence $\{v_r\}$ of natural numbers and the sequence $\{p_r\}$ of positive real numbers are chosen such that the relations

(8) $$\sum_{r=1}^{\infty} p_r v_r < \infty,$$

and

(9) $$\lim_{r \to \infty} p_r v_r \log v_r = \infty$$

are satisfied. An appropriate choice is for example

$$v_r = 2^{r^3} \quad \text{and} \quad p_r = \frac{1}{r^2 v_r} \qquad (r=1, 2, \ldots).$$

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[2]) An orthonormal system $\{\varphi_k(x)\}$ is called a convergence system if every series $\Sigma a_k \varphi_k(x)$ whose coefficients satisfy the condition (2) is convergent almost everywhere.

8. By (8) we can easily see that

$$\sum_{k=0}^{\infty} c_k^2 = \sum_{r=1}^{\infty} \sum_{k=N_{r-1}+1}^{N_r} c_k^2 = 2 \sum_{r=1}^{\infty} p_r^2 v_r^2 \leqq C_5 \sum_{r=1}^{\infty} p_r v_r < \infty.$$

We show that (8) implies also the convergence of the partial sums $\{s_{N_r}(x)\}$ and $\{s_{N_r'}(x)\}$ of the series (3) almost everywhere. On account of

$$\sum_{r=1}^{\infty} \int_0^1 |s_{N_r}(x) - s_{N_{r-1}}(x)| dx \leqq \sum_{r=1}^{\infty} \left\{ \int_0^1 (s_{N_r}(x) - s_{N_{r-1}}(x))^2 dx \right\}^{\frac{1}{2}} =$$

$$= \sum_{r=1}^{\infty} \left\{ \sum_{k=N_{r-1}+1}^{N_r} c_k^2 \right\}^{\frac{1}{2}} = \sqrt{2} \sum_{r=1}^{\infty} p_r v_r < \infty,$$

we infer, by applying the theorem of B. Levi, that the sequence $\{s_{N_r}(x)\}$ is convergent. The convergence of $\{s_{N_r'}(x)\}$ almost everywhere follows in the same way.

9. Now we are able to prove the T-summability of the series (3) almost everywhere. On the one hand, if $N_r' < i \leqq N_r$, then we have

$$t_i(x) = s_{N_r}(x);$$

on the other hand, if $N_{r-1} < i \leqq N_r'$, then

$$t_i(x) = \frac{1}{2} s_i(x) + \frac{1}{2} s_{i+v_r^2}(x) = \frac{1}{2} s_{N_{r-1}}(x) + \frac{1}{2} s_{N_r'}(x) + \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Phi_k(x).$$

For the sake of brevity, we write

$$R(r, i; x) = \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Phi_k(x).$$

For our purpose it is enough to show that $R(r, i; x)$ tends to 0 almost everywhere in (0, 1) as $r \to \infty$. Taking into account the definition of the coefficients c_k and (5), we can see that the $R(r, i; x)$ equals 0 at every point $x \in G_r(2)$. In case $x \in G_r'(1) \cup \cup G_r''(1)$, we get by a simple calculation that

$$\Phi_k(x) = \frac{\sqrt{2+C}}{\sqrt{2}} \Psi_k(x) \quad \text{if} \quad x \in G_r'(1), \qquad \Phi_k(x) = -\frac{\sqrt{2+C}}{\sqrt{2}} \Psi_k(x) \quad \text{if} \quad x \in G_r''(1)$$

$(N_{r-1} < k \leqq N_r', \ r = 1, 2, \cdots)$. Hence we can write

$$R(r, i; x) = \pm \frac{\sqrt{2+C}}{\sqrt{2}} \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Psi_k(x),$$

according as $x \in G_r'(1)$ or $x \in G_r''(1)$. Applying Lemma 2, we infer that $R(r, i; x)$ tends to 0 almost everywhere as $r \to \infty$.

10. To accomplish the proof, we have to show that (4) is also satisfied. Let us consider the sets $G_r(3)$ $(r=1, 2, \cdots)$. According to the definition of the intervals I'_σ, I''_σ $(\sigma=1, 2, \cdots, s)$ and $G_r(3)$, we can easily see that the sets $G_r(3)$ are stochastically independent. Applying the Borel—Cantelli lemma we get, by virtue of (7),

$$\left|\overline{\lim_{r\to\infty}}\, G_r(3)\right| = 1. \tag{10}$$

Let $N_{r-1} < i \leqq N_r$. By looking at the inequality

$$|a-b|^\gamma \geqq C(\gamma)|a|^\gamma - |b|^\gamma \qquad (\gamma > 0),$$ [3]

where $C(\gamma)$ denotes a positive constant depending only on γ, we obtain the estimate

$$\sum_{k=0}^{\infty} \alpha_{ik}|s_k(x)-f(x)|^\gamma = \frac{1}{2}|s_i(x)-f(x)|^\gamma + \frac{1}{2}|s_{i+v_r^2}(x)-f(x)|^\gamma \geqq$$

$$\geqq C(\gamma)\left\{\left|\sum_{k=N_{r-1}+1}^{i} c_k\Phi_k(x)\right|^\gamma + \left|\sum_{k=N_r'+1}^{i+v_r^2} c_k\Phi_k(x)\right|^\gamma\right\} -$$

$$-\frac{1}{2}|s_{N_{r-1}}(x)|^\gamma - \frac{1}{2}|s_{N_r'}(x)|^\gamma - |f(x)|^\gamma.$$

By virtue of (6), there exists an index $i=l(x)$ $\left(N'_{r-1} < l(x) \leqq N'_r\right)$ for almost every point $x \in G_r(3)$ such that

$$\sum_{k=0}^{\infty} \alpha_{l(x),k}|s_k(x)-f(x)|^\gamma \geqq C_4 C(\gamma) p_r v_r \log v_r - C(x)$$

holds, where $C(x)$ is a positive constant depending only on x. Here we again took into consideration that the sequences $\{s_{N_r}(x)\}$ and $\{s_{N_r'}(x)\}$ converge almost everywhere. By (10) this estimate holds at almost every point $x \in (0, 1)$ for infinitely many values of r. Using (9), we get that the relation (4) is satisfied almost everywhere.

We have thus completed the proof of our theorem.

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[3]) If $0 < \gamma \leqq 1$ then this inequality follows from $|a+b|^\gamma \leqq |a|^\gamma + |b|^\gamma$, and if $\gamma > 1$ then it follows from $|a+b|^\gamma \leqq 2^{\gamma-1}(|a|^\gamma + |b|^\gamma)$.

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