# A note on the existence of derivations

By R. J. LOY*) in Clayton (Victoria, Australia)

Let $K$ be a commutative ring with unit; $\mathfrak{A}$, $\mathfrak{B}$ two $K$-algebras, and denote by $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ the (left) $K$-module of all $K$-linear mappings of $\mathfrak{A}$ into $\mathfrak{B}$. We write $\mathfrak{L}(\mathfrak{A})$ for $\mathfrak{L}(\mathfrak{A}, \mathfrak{A})$ and note that this is a $K$-algebra. If $\varphi$, $D \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ and $D$ satisfies the equation

$$D(xy) = Dx \cdot \varphi y + \varphi x \cdot Dy$$

for all $x, y \in \mathfrak{A}$, we call $D$ a $\varphi$-derivation. If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and $I \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ is the identity map on $\mathfrak{A}$, an $I$-derivation will be termed a derivation. We will only consider $\varphi$-derivations where $\varphi$ is a homomorphism.

Note that our $\varphi$-derivations are different from the $\varphi$-derivations of AMITSUR [1] and JACOBSON [3], which are additive mappings defined on fields satisfying $D(xy) = \varphi x \cdot Dy + Dx \cdot y$ (or $D(xy) = Dx \cdot \varphi y + x \cdot Dy$) for all $x$, $y$ in the field. Our $\varphi$-derivations satisfy the defining equation for the $(\varphi, \varphi)$- derivations of [4], p. 177; they are closely related to the $\varphi$-derivations of order one of [5].

Now let $D \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ be a $\varphi$-derivation of $\mathfrak{A}$ into $\mathfrak{B}$ for some homomorphism $\varphi \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$. We define the 'spread' $\mathfrak{S}$ of the $D$ as the smallest subalgebra of $\mathfrak{B}$ containing both the range of $D$ and the range of $\varphi$. $\mathfrak{S}$ is in fact the smallest subalgebra of $\mathfrak{B}$ such that both $\varphi$ and $D$ are in $\mathfrak{L}(\mathfrak{A}, \mathfrak{S})$, that is, such that $D$ is a $\varphi$-derivation from $\mathfrak{A}$ into $\mathfrak{S}$. It is clear that $\mathfrak{S}$ depends on the mapping $\varphi$, and if $D$ is a $\varphi$-derivation for more than one $\varphi$ it may have more than one spread. In the cases where we use this notion of spread however, it will be clear which mapping $\varphi$ is being considered, and no explicit mention of it will be made.

We will denote the algebra direct sum of two $K$-algebras $\mathfrak{A}, \mathfrak{B}$ by $\mathfrak{A} \oplus \mathfrak{B}$, and their $K$-module direct sum by $\mathfrak{A} + \mathfrak{B}$. Thus $\mathfrak{A} \oplus \mathfrak{B}$ is the $K$-algebra of all pairs $(a, b)$ with $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and componentwise operations, while $\mathfrak{A} + \mathfrak{B}$ is the $K$-module of all such pairs and will not be assumed to have the algebra structure of $\mathfrak{A} \oplus \mathfrak{B}$.

*) The author is a General Motors—Holden's Research Fellow.

If $\varphi \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ is a homomorphism, the $K$-linear isomorphism $\Phi$ of $\mathfrak{A}$ into $\mathfrak{A} \oplus \mathfrak{B}$ defined by

$$\Phi x = (x, \varphi x)$$

for all $x \in \mathfrak{A}$, will be called the $\varphi$-embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{B}$. Any isomorphism of this type will be termed an embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{B}$.

With these notations we have the following result.

Theorem 1. *Let $\mathfrak{A}$ be a K-algebra with a (K-module) direct sum decomposition $\mathfrak{A} = \mathfrak{B} + \mathfrak{J}$ where $\mathfrak{J}$ is a proper left ideal of $\mathfrak{A}$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ but not a left ideal. Then there is an embedding $\Phi$ of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$, such that $\mathfrak{A}$ admits a nonzero $\Phi$-derivation $D: \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ satisfying $D(\mathfrak{B}) = \{0\}$.*

Proof. For $a \in \mathfrak{A}$, let $a = x + y$ be the unique decomposition of $a$ with $x \in \mathfrak{B}$, $y \in \mathfrak{J}$, and define idempotent mappings $P, Q \in \mathfrak{L}(\mathfrak{A})$ by $Pa = x$, $Qa = y$. Then we have immediately that $P^2 = P$, $PQ = QP = 0$, $Q^2 = Q$ and $P + Q = I$.

For $a \in \mathfrak{A}$ denote by $\tilde{a}$ the image of $a$ in $\mathfrak{L}(\mathfrak{A})$ under the left regular representation and define $K$-linear mappings $\varphi, \Delta$ from $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{A})$ by $\varphi a = P\tilde{a}P + Q\tilde{a}Q$ and $\Delta a = Q\tilde{a}P$ for all $a \in \mathfrak{A}$. Since $\mathfrak{J}$ is a left ideal it is invariant under each $\tilde{a}$ for $a \in \mathfrak{A}$ and so $P\tilde{a}Q = 0$ for any $a \in \mathfrak{A}$. But then if $x, y \in \mathfrak{A}$, $\widetilde{(xy)} = \tilde{x}\,\tilde{y}$ and so

$$\varphi(xy) = P\tilde{x}(P+Q)\tilde{y}P + Q\tilde{x}(P+Q)\tilde{y}Q = P\tilde{x}P\tilde{y}P + Q\tilde{x}Q\tilde{y}Q =$$
$$= (P\tilde{x}P + Q\tilde{x}Q)(P\tilde{y}P + Q\tilde{y}Q) = \varphi x \cdot \varphi y.$$

Also

$$\Delta(xy) = Q\tilde{x}(P+Q)\tilde{y}P = Q\tilde{x}P\tilde{y}P + Q\tilde{x}Q\tilde{y}P =$$
$$= Q\tilde{x}P(P+Q)\tilde{y}(P+Q) + (P+Q)\tilde{x}(P+Q)Q\tilde{y}P = \Delta x \cdot \varphi y + \varphi x \cdot \Delta y.$$

Thus $\varphi$ is a homomorphism and $\Delta$ is a $\varphi$-derivation, which furthermore is non-zero since $\mathfrak{B}$ is not a left ideal.

We now make use of a construction of Singer and Wermer [6]. Let $\Phi$ be the $\varphi$-embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ and define a $K$-linear mapping $D: \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ by $Da = (0, \Delta a)$ for all $a \in \mathfrak{A}$. It is easily seen that $D$ is a $\Phi$-derivation, non-zero since $\Delta$ is non-zero. If $x \in \mathfrak{B}$ then $(\tilde{x}P)y \in \mathfrak{B}$ for any $y \in \mathfrak{A}$, so that $(Q\tilde{x}P)y = 0$. Thus $\Delta x = 0$ if $x \in \mathfrak{B}$, that is, $D(\mathfrak{B}) = \{0\}$.

The reason for wanting $\Phi$ to be an isomorphism is that we can identify $\mathfrak{A}$ with $\Phi(\mathfrak{A})$ to get the following result.

Corollary 1. *Let $\mathfrak{A}$ satisfy the conditions of the theorem. Then $\mathfrak{A}$ admits a non-zero derivation D into an extension of $\mathfrak{A}$ such that $D(\mathfrak{B}) = \{0\}$.*

The spread of (the $\varphi$-derivation) $D$ is in general non-commutative even when $\mathfrak{A}$ is commutative [1]). A necessary and sufficient condition for the spread to be commutative is given by the following result.

[1]) The range of $D$ is easily seen to be a zero ring.

Theorem 2. *If $\mathfrak{A}$ is commutative then the spread of $D$ is commutative if and only if $\mathfrak{J}^2\mathfrak{B}=0$.*

Proof. From the definition of $D$, the spread of $D$ is commutative if and only if the spread of $\Delta$ is commutative. Now the spread of $\Delta$ is the algebra generated by the set $\{\varphi a, \Delta a: a\in\mathfrak{A}\}$. Since $\mathfrak{A}$ is commutative and $\varphi$ is a homomorphism $\varphi x\cdot\varphi y=\varphi y\cdot\varphi x$ for all $x, y\in\mathfrak{A}$. Also, from the definitions, $\Delta x\cdot\Delta y=\Delta y\cdot\Delta x=0$. Thus it suffices to consider necessary and sufficient conditions for $\Delta x\cdot\varphi y=\varphi y\cdot\Delta x$, that is, $Q\tilde{x}P\tilde{y}P=Q\tilde{y}Q\tilde{x}P$ for all $x, y\in\mathfrak{A}$.

(i) Necessity of stated condition. Let $x, y\in\mathfrak{J}$, $z\in\mathfrak{B}$. Then $(Q\tilde{x}P\tilde{y}P)z=(Q\tilde{x}P\tilde{y})z=0$ since $\tilde{y}(z)=yz\in\mathfrak{J}$ and $P(\mathfrak{J})=\{0\}$. On the other hand, $(Q\tilde{y}Q\tilde{x}P)z=(Q\tilde{y}Q\tilde{x})z=(Q\tilde{y}Q)xz=(Q\tilde{y})xz=Q(yxz)=yxz$. Thus the condition is necessary.

(ii) Sufficiency of stated condition. If $z\in\mathfrak{J}$ then $(Q\tilde{x}P\tilde{y}P)z=0=(Q\tilde{y}Q\tilde{x}P)z$ for any $x, y\in\mathfrak{A}$ since $Pz=0$. Since $\mathfrak{A}=\mathfrak{B}+\mathfrak{J}$ it thus suffices to consider $z\in\mathfrak{B}$. Thus suppose $z\in\mathfrak{B}$, and let $x, y\in\mathfrak{A}$ have decompositions $x=u_1+v_1$; $y=u_2+v_2$; $u_i\in\mathfrak{B}, v_i\in\mathfrak{J}$ for $i=1,2$. Then the decompositions of $xz, yz, xu_2z$ are $u_1z+v_1z$, $u_2z+v_2z$, $u_1u_2z+v_1u_2z$, respectively. Thus $(Q\tilde{x}P\tilde{y}P)z=(Q\tilde{x}P)(u_2z+v_2z)=Q(xu_2z)=v_1u_2z$, and $(Q\tilde{y}Q\tilde{x}P)z=(Q\tilde{y}Q)(u_1z+v_1z)=Q(yv_1z)=yv_1z$. The difference between these is $yv_1z-v_1u_2z=v_1z(y-u_2)=v_1v_2z=0$, since $v_1, v_2\in\mathfrak{J}$, $z\in\mathfrak{B}$. Thus the condition is sufficient.

It follows that the extension of $\mathfrak{A}$ in Corollary 1 may be taken to be commutative if $\mathfrak{A}$ is commutative and $\mathfrak{J}^2\mathfrak{B}=0$.

Corollary 2. *Let $\mathfrak{A}$ be a commutative K-algebra with $\mathfrak{A}^2=\mathfrak{A}$.*[2] *Suppose that $\mathfrak{A}$ has a direct sum decomposition $\mathfrak{A}=\mathfrak{B}+\mathfrak{N}$ where $\mathfrak{B}$ is a subalgebra and $\mathfrak{N}$ is a nontrivial nilpotent ideal. Then $\mathfrak{A}$ admits a non-zero derivation into an extension algebra of $\mathfrak{A}$ which annihilates $\mathfrak{B}$. If $\mathfrak{N}$ is a zero ring or if the sum is an algebraic direct sum then the extension algebra may be taken to be commutative.*

Proof. By hypothesis $\mathfrak{B}$ is a subalgebra, so using Corollary 1 it suffices to show that it is not an ideal. Supposing to the contrary, we have $\mathfrak{B}\mathfrak{N}\subseteqq\mathfrak{B}$ and so $\mathfrak{A}^2\subseteqq\mathfrak{B}+\mathfrak{N}^2$ whence $\mathfrak{N}^2=\mathfrak{N}$. But this is impossible since $\mathfrak{N}$ is non-trivial and nilpotent.

The last statement is clear from Theorem 2.

*Acknowledgements* The idea of Theorem 1 stemmed from a perusal of the matrix proof of the theorem of MASCHKE in the theory of group representations. See, for example, Theorem 16. 3. 1 of [2].

The author would like to thank Professor J. B. MILLER for helpful discussions concerned with this paper.

---

[2]) This is true, for instance, if $\mathfrak{A}$ has an identity.

## References

[1] S. A. Amitsur, A generalization of a theorem on linear differential equations, *Bull. Amer. Math. Soc.*, **54** (1948), 937—941.
[2] M. Hall, *The theory of groups* (New York, 1965).
[3] N. Jacobson, Pseudolinear transformations, *Ann. of Math.*, **38** (1937), 484—507.
[4] ——— *Lie algebras* (New York, 1962).
[5] H. Osborn, Modules of differentials. I, *Math. Annalen*, **170** (1967), 221—244.
[6] I. Singer and J. Wermer, Derivations on commutative normed algebras, *Math. Annalen*, **129** (1955), 260—264.

*(Received February 20, 1968)*

MONASH UNIVERSITY
CLAYTON, VICTORIA
AUSTRALIA