Cardinals inaccessible with respect to a function defined on pairs of cardinals

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In the present paper we are going to prove a lemma based on the theory of stationary classes with the aid of which a formula can be derived for the cofinality number of an arbitrary cardinal. Replacing the particular function occurring on the right hand side of this formula by a function variable we are led to a generalization of cofinality and thus at last we shall get a generalization of algebraic type of the notion of inaccessible cardinals. A simple by-product of our investigations will be that in a sense almost every weakly inaccessible cardinal is strongly inaccessible too. Our lemma might be of some interest in itself as well.

Notation. In the sequel Greek letters will always denote cardinal numbers and the class of all cardinals will be denoted by C. The least ordinal exceeding a class H of cardinal numbers will be denoted by $\sup H$. This is a cardinal number unless H is a proper class; in this latter case $\sup H = On$, On denoting the class of all ordinal numbers.

1. Stationary classes[1])

Here we give a brief sketch of the most important results in the theory of stationary sets used below. We do not deal with the generalized form of the theory as given by G. FODOR and A. HAJNAL [1]; however this theory might be most illuminating in the understanding of the special theory as well.

Where the adjective "closed" if used for a subclass of C is meant in the topology induced by the natural ordering of C, a subclass of C is said to be stationary if it meets every closed proper class contained in C. One of the most important results for stationary classes is the following one (see [2], Hilfssatz).

Theorem 1.1. *Suppose that $\{S_\alpha\}_{\alpha\in H}$ $(H \subseteqq C)$ is a sequence of non-empty and non-stationary classes and the class $\{\sigma_\alpha\}_{\alpha\in H}$ of their first elements, which are*

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[1]) A more detailed account of the subject presented here will be given in the authors' forthcoming book on stationary classes, regressive functions and their applications.

assumed to be mutually distinct, is not stationary either. Then the union class $\bigcup_{\alpha \in H} S_\alpha$ *is also non-stationary.*

However for later use it seems preferable to formulate this results in terms of a regressive function by which we mean a mapping f of a subclass of C into C statisfying $f(\alpha) < \alpha$. The equivalence of the next result to Theorem 1. 1 is rather obvious (see [2], Satz 2).

Theorem 1. 2. *If the regressive function f is defined on a stationary class then there exists a cardinal μ such that the class $\{\xi : f(\xi) = \mu\}$, ξ running over the domain of f, is stationary too.*

Now we derive from this last result a corollary which is already of special interest in order to achieve the proof of our main lemma mentioned below.

Corollary 1. 3. *Let $h(\lambda, \xi)$ an arbitrary mapping of $C \times C$ into C. Then the class*

$$S = \{\alpha : \exists (\lambda, \xi) (\lambda, \zeta < \alpha \,.\, \& \,.\, h(\lambda, \xi) \geqq \alpha)\}$$

is not stationary.

Proof. Assuming S to be stationary, by the previous theorem we obtain the existence of a cardinal λ_0 and of a stationary subclass S' of S such that for any $\alpha \in S'$ we have $\lambda_0 < \alpha$ and

$$(\exists \xi)(\xi < \alpha \,.\, \& \,.\, h(\lambda_0, \xi) > \alpha);$$

so by a repeated application of the preceding theorem we have that there exists a cardinal ξ_0 and a stationary class $S'' \subseteqq S'$ such that $\alpha \in S''$ implies $\xi_0 < \alpha$ and $h(\lambda, \xi_0) \geqq \alpha$; so S'' is not cofinal to C in contradiction to its stationarity.

2. The main lemma

In the sequel $h(\alpha, \xi)$ denotes a mapping of $C \times C$ into C satisfying

$$\sup_\alpha h(\alpha, \xi) = On \tag{2. 1}$$

whichever the cardinal ξ may be.

We start by proving the following

Lemma 4. 1. *Define the classes*

$$P(\alpha) = \{\xi : \xi < \alpha \,.\, \& \,.\, h(\alpha, \xi) > \alpha\},$$

$$Q(\alpha) = \{\xi : \xi < \alpha \,.\, \& \,.\, h(\alpha, \xi) > \sup_{\eta < \alpha} h(\eta, \xi)\},$$

depending on the arbitrary cardinal α. Then we have $P(\alpha) = Q(\alpha)$ for almost all α, meaning by this latter expression that the exceptional α's form a non-stationary class.

Proof. (i) Assume that

$$P(\alpha) \ni \xi \notin Q(\alpha).$$

Then $\xi < \alpha$ and $\alpha < h(\alpha, \xi) \leqq \sup_{\eta < \alpha} h(\eta, \xi)$, i.e. for some $\eta < \alpha$ we have $h(\eta, \xi) > \alpha$. Because of the inequality $\eta, \xi < \alpha$ the class of α's of this kind is not stationary, according to Corollary 1. 3.

(ii) Suppose that for a stationary class S of α's we have

$$P(\alpha) \not\ni \xi_\alpha \in Q(\alpha).$$

Since $\xi_\alpha < \alpha$, by Theorem 3. 4 we obtain the existence of a cardinal ξ_0 and of a stationary class $S' \subseteqq S$ such that for $\alpha \in S'$ we have $\xi_\alpha = \xi_0$. So

$$\sup_{\eta < \alpha} h(\eta, \xi_0) < h(\alpha, \xi_0) \leqq \alpha.$$

Thus there exists a β_0 and a stationary subclass S'' of S' such that for $\alpha \in S''$ we have

$$\sup_{\eta < \alpha} h(\eta, \xi_0) = \beta_0.$$

Since S'' is stationary it is cofinal to C, so our latest equality implies

$$\sup_{\eta} h(\eta, \xi_0) = \beta_0 < On,$$

contradicting (2. 1).

Below the notation $\min H$ will indicate the first element of the class H. In case the class H is empty, then in a natural way we put $\min H = On$. The next lemma, which we might call our main lemma, can be easily derived from our preceding lemma, so the proof will not be carried out.

Lemma 2. 2. *Let*

$$p(\alpha) = \min \{\xi : h(\alpha, \xi) > \alpha\},$$

$$q(\alpha) = \min \{\xi : h(\alpha, \xi) > \sup_{\eta < \alpha} h(\eta, \alpha)\}.$$

Then for almost all α *either* $p(\alpha) = q(\alpha)$ *or* $p(\alpha), q(\alpha) \geqq \alpha$ *holds.*

Corollary 2. 3. *If* α^* *denotes the least cardinal cofinal to* α, *then*

$$\alpha^* = \min \{\xi : \alpha^\xi > \alpha\}$$

holds for almost all α.

Proof. As is easy to derive from a classical result of J. König [3], we have $\alpha^{\alpha^*} > \alpha$. Thus putting $h(\alpha, \xi) = \alpha^\xi$, we obtain

$$p(\alpha) \leqq \alpha^*. \tag{2. 2}$$

On the other hand we have

$$q(\alpha) \geqq \alpha^*, \tag{2. 3}$$

as a consequence of the almost obvious equality

$$\alpha^{\xi} = \sup_{\eta<\alpha^*} \eta^{\xi},$$

being valid for every $\xi<\alpha^*$.

Making use of Lemma 2. 2 and the inequalities (2. 2) and (2. 3) we obtain that $p(\alpha)=\alpha^*$ holds for almost all α, which was to be proved.

3. Inaccessible cardinals

Now we recall two well-known definitions and will indicate heuristically the way which leads us to their generalizations.

Definition 3. 1. The cardinal number α is weakly inaccessible if it is regular (i. e. $\alpha^*=\alpha$) and moreover $\xi^+<\alpha$ for each $\xi<\alpha$, ξ^+ denoting the least cardinal number exceeding ξ. Thus denoting by I the class of all weakly inaccessible cardinals, our definition may be written in a more formal way:

$$I = \{\alpha:\alpha^* = \alpha . \& . (\forall \xi)(\xi<\alpha . \rightarrow . \xi^+<\alpha)\}.$$

Definition 3. 2. The cardinal number α is strongly inaccessible, otherwise said $\alpha\in J$, if $\alpha\in I$ and for any $\lambda, \xi<\alpha$ the inequality $\lambda^{\xi}<\alpha$ holds. Formally

$$J = \{\alpha\in I:(\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^{\xi}<\alpha)\}.$$

According to Corollary 2. 3 we can replace α^* by $\min\{\xi: \alpha^{\xi}\leqq\alpha\}$ for almost all α. Then the condition $\alpha^*=\alpha$ in Definition 3. 1 turns into the one

$$(\forall \xi)(\xi<\alpha . \rightarrow . \alpha^{\xi}\leqq\alpha).$$

So if we define the class

$$I' = \{\alpha:(\forall \xi)(\xi<\alpha . \rightarrow . \alpha^{\xi^+}\leqq\alpha\},$$

it is easily seen that the classes I' and I are almost equal, i. e. their symmetrical difference $I'\triangle I$ is not stationary. Thus it is obvious that for

$$J' = \{\alpha\in I':(\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^{\xi}<\alpha)\} = \{\alpha\in I':(\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^{\xi^+}<\alpha)\}$$

the class $J'\triangle J$ is not stationary either.

As seen in the definitions of I' and J' the function λ^{ξ^+} is crucial there which, replaced by an arbitrary mapping $h(\lambda, \xi)$ of $C\times C$ into C, allows us to generalize the above concepts in a suitable way:

Definition 3. 3. $\alpha\in I_h$ i. e. α is weakly h-inaccessible if $h(\alpha, \xi)\leqq\alpha$ for each $\xi<\alpha$. Formally $I_h = \{\alpha:(\forall \xi)(\xi<\alpha . \rightarrow . h(\alpha, \xi)\leqq\alpha)\}$.

Definition 3. 4. $\alpha\in J_h$ i. e. α is strongly h-inaccessible, if $\alpha\in I_h$ and for each $\lambda, \xi<\alpha$ we have $h(\lambda, \xi)<\alpha$. Formally $J_h = \{\alpha\in I_h:(\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . h(\lambda, \xi)<\alpha)\}$.

These two definitions make it clear that there is no essential difference between weakly and strongly inaccessible cardinals. More precisely, on account of Corollary 1. 3 we have

Theorem 3. 5. *The class $I_h - J_h$ is not stationary.*

Restating this result in the particular case $h(\alpha, \xi) = \alpha^{\xi^+}$ and taking into consideration that the symmetrical differences $I' \triangle I$ and $J' \triangle J$ are non-stationary, we get that the class $I - J$ is not so either, i. e. *almost all weakly inaccessible cardinals are strongly inaccessible too.*

References

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[3] J. KÖNIG, Zum Kontinuumproblem, *Math. Ann.*, **65** (1905), 177.

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