On semigroups which are semilattices of groups

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Let S be a semigroup. [1]) We shall say that S has property (M) if the relation

$$L \cap R = LR \tag{1}$$

holds for each left ideal L and for each right ideal R of S. Furthermore we say that S has property (L) resp. (R) if the relation

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

resp.

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

holds for every pair of left and right ideals of S, respectively.

Recently the author proved that a semigroup S having both properties (L) and (R) is a disjoint union of groups. In this note we prove that a semigroup S with property (M) is a semilattice of groups. It may also be proved that a semigroup having both properties (L) and (R) is a semilattice of groups.

First we prove some results concerning a semigroup having the property (M).

Theorem 1. *In a semigroup S having property (M) every one-sided ideal is a two-sided ideal.*

Proof. Let S be a semigroup with property (M), and suppose that L is an arbitrary left ideal of S. Then by (1) we have

$$L = L \cap S = LS$$

that is, L is also a right ideal of S. Therefore L is a two-sided ideal of S, as we stated. Similarly, any right ideal R of S is also a two-sided ideal, because of

$$R = S \cap R = SR.$$

Theorem 2. *Any semigroup with property (M) is a normal semigroup.*

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[1]) We adopt the terminology of CLIFFORD and PRESTON [2]. See also LJAPIN [6] and SZÁSZ [8].

Proof. Let S be a semigroup having the property (M), and a be an arbitrary element of S. Then (1) implies that

(4) $$aS = S \cap aS = SaS.$$

Similarly,

(5) $$Sa = Sa \cap S = SaS.$$

The relations (4) and (5) imply

(6) $$aS = Sa$$

for each element a in S, i.e., S is a normal semigroup. (See [7]).

Theorem 3. *Any semigroup S having property (M) is regular.* [2])

Proof. Let S be a semigroup with property (M). In a recent note [4] the author proved that a normal semigroup is regular if and only if every left ideal L of S is idempotent, i.e. $L^2 = L$. By Theorem 2, S is normal. If A is a left ideal of S then by Theorem 1, A is a two-sided ideal of S, and the relation (1) implies that

$$A = AA$$

in case of $L = R = A$. Therefore any ideal of S is idempotent, which proves the regularity of S.

Now we are ready to prove the following result:

Theorem 4. *Any semigroup S with property (M) is a semilattice of groups.*

Proof. Let S be a semigroup having the property (M). First we show that for every element a of S

(7) $$a \in Sa^2 \cap a^2 S$$

holds. By Theorem 3 we have $a \in aSa$. But by Theorem 2, the semigroup S is normal, i.e. $aS = Sa$ for each element a in S. Hence $aSa = a^2 S = Sa^2$. Therefore the inclusion (7) follows.

Secondly we show that the idempotent elements of S commute, that is, if e and f are idempotent elements of S, then

(8) $$ef = fe$$

holds. But this is an easy consequence of a result due to Schwarz [7] in virtue of which the idempotent elements of a normal semigroup lie in the center.

Now (7) and (8) imply that S is a semilattice of groups, by a result of Croisot [3] (see also Clifford [1], Theorem 8).

The following result may be proved analogously:

[2]) More generally, *a semigroup S having property (M) is an inverse semigroup.* This follows from Theorems 2 and 3.

Theorem 5. *Suppose that the semigroup S has properties (L) and (R). Then S is a semilattice of groups.*

Corollary. *If S is a semigroup having both properties (L) and (R) then S is a union of disjoint groups.*

This is an easy consequence of Theorem 5, and was recently proved by the author in [5].

References

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