On power series with positive coefficients

By RAIS SHAH KHAN in Aligarh (India)

In this note we prove:

Theorem. *Let* $F(x)=\sum_{k=0}^{\infty} a_k x^k,\ a_k \geqq 0,\ 0 \leqq x<1,\ S_n=\sum_{k=0}^{n} a_k,$ *and* $r<1$. *Then, for* $1 \leqq p \leqq \infty$,

$$\left(\int_0^1 (1-x)^{-r}(F(x))^p\,dx\right)^{1/p}<\infty$$

if and only if

$$\left(\sum_{n=1}^{\infty} n^{r-2} S_n^p\right)^{1/p}<\infty.^{1)}$$

This theorem reduces for $r=0$ to a theorem of ASKEY [1] and for $p=1$ to a theorem of HEYWOOD [2]. The method of proof follows that of ASKEY.

Proof. We may assume $1<p<\infty$, as the two limit cases are trivial.

Necessity. We write $1-x=y$. Then by virtue of the fact that $\left(1-\frac{1}{n}\right)^n$ $(n=1,2,\ldots)$ is an increasing sequence, we have for $\frac{1}{n+1} \leqq y \leqq \frac{1}{n},\ n \geqq 2$

$$F(1-y) \geqq \sum_{k=0}^{n} a_k(1-y)^k \geqq \left(1-\frac{1}{n}\right)^n \sum_{k=0}^{n} a_k,$$

i.e.

$$F(1-y) \geqq A S_n \quad \text{for} \quad \frac{1}{n+1} \leqq y \leqq \frac{1}{n} \qquad (n=2,3,\ldots),$$

---

[1]) For $p=\infty$ one has to take the limits as $p \to \infty$, i.e. $\sup_{0 \leqq x<1} F(x)$ and $\sup_{1 \leqq n<\infty} S_n$.

where A is a positive constant not necessarily the same at each occurrence. Thus

$$\sum_{n=1}^{\infty} n^{r-2} S_n^p \leqq A \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-r} S_n^p \, dy = A \int_{\frac{1}{2}}^{1} y^{-r} S_1^p \, dy + A \sum_{n=2}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-r} S_n^p \, dy \leqq$$

$$\leqq A + A \int_0^{\frac{1}{2}} y^{-r} (F(1-y))^p \, dy \leqq A + A \int_0^1 (1-x)^{-r} (F(x))^p \, dx < \infty.$$

Sufficiency. We have

$$\int_0^1 (1-x)^{-r} (F(x))^p \, dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-r} (F(1-y))^p \, dy =$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-r} \left(\sum_{k=0}^{\infty} a_k (1-y)^k \right)^p dy \leqq A \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-r} \left(\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right)^p dy \leqq$$

$$\leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right)^p \leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{j=0}^{\infty} \sum_{k=nj}^{n(j+1)} a_k \left(1 - \frac{1}{n+1}\right)^k \right)^p \leqq$$

$$\leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{j=0}^{\infty} \left(1 - \frac{1}{n+1}\right)^{nj} \sum_{k=nj}^{n(j+1)} a_k \right)^p \equiv \Sigma_1.$$

Since $\frac{1}{2} \geqq \left(1 - \frac{1}{n+1}\right)^n \geqq \left(1 - \frac{1}{n+2}\right)^{n+1}$ for $n = 1, 2, \ldots$; we obtain that

$$\Sigma_1 \leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{j=0}^{\infty} 2^{-j} \sum_{k=0}^{n(j+1)} a_k \right)^p \leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{i=1}^{\infty} 2^{-\frac{i}{2}} \cdot 2^{-\frac{i}{2}} \sum_{k=0}^{ni} a_k \right)^p \leqq$$

$$\leqq A \sum_{n=1}^{\infty} n^{r-2} \left(\sum_{i=1}^{\infty} 2^{-\frac{ip'}{2}} \right)^{\frac{p}{p'}} \left(\sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} S_{ni}^p \right) \leqq A \sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} S_{ni}^p =$$

$$= A \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} \sum_{n=1}^{\infty} n^{r-2} S_{ni}^p = A \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} \cdot i^{-r+2} \sum_{n=1}^{\infty} (in)^{r-2} S_{ni}^p \leqq A \sum_{n=1}^{\infty} n^{r-2} S_n^p < \infty.$$

The proof is thus completed.

I wish to express my gratitude to Dr. S. M. Mazhar for his valuable guidance.

Added in proof. The author has recently observed that the theorem remains true also for $0<p<1$.

References

[1] R. ASKEY, L^p behavior of power series with positive coefficients, *Proc. Amer. Math. Soc.*, **19** (1968), 303—305.

[2] P. HEYWOOD, Integrability theorems for power series and Laplace transforms. I, *J. London Math. Soc.*, **30** (1955), 302—310.

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

(Received July 1, 1968)