On a problem of P. Erdős

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In this note we are going to prove the following

Theorem. *Suppose that for an additive number theoretic function*[1] $f(n)$ *the difference* $f(n+1)-f(n)$ *is bounded. Then there exists a decomposition* $g(n)+h(n)$ *of* $f(n)$ *wherein* $g(n)$ *is completely additive*[2] *and* $h(n)$ *is bounded.*

The uniqueness of this decomposition is most obvious. In fact, assuming its existence we have

$$g(n) = \lim_{t\to\infty} \frac{f(n^t)}{t} \quad \text{and} \quad h(n) = f(n)-g(n).$$

The above theorem was conjectured by P. ERDŐS [1], p. 3, in a stronger form where he asserts that in the above decomposition $g(n)$ must be a constant multiple of $\log n$. (For the restating of the problem see [2], p. 6 and [3], p. 162.) As yet we have not been able to overcome the difficulties in deciding whether or not this latter assertion is true.

In order to prove the above theorem we show the existence of the limit $g(n) = \lim\limits_{t\to\infty} \dfrac{f(n^t)}{t}$ and prove its coincidence with $f(n)$ apart from a bounded term. First we prove three lemmas in which M will be a bound for the expression $|f(n+1)-f(n)|$.

Lemma 1. *If s is prime to $n-1$ then*

$$|f(n^s)-sf(n)| \leqq 2sM.$$

Proof. As is easily seen

$$s = \sum_{i=0}^{s-1} 1 \equiv \sum_{i=0}^{s-1} n^i \pmod{n-1};$$

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[1]) I.e. for a function defined on the set of positive integers satisfying the relation $f(ab) = f(a)+f(b)$ whenever a is prime to b.

[2]) I.e. $g(ab) = g(a)+g(b)$ for any two positive integers a and b.

thus by the assumptions of the lemma we have that $\sum_{i=0}^{s-1} n^i$ is prime to $n-1$ as well. So an easy calculation shows (note $f(1)=0$):

$$|f(n^s)-sf(n)| \leqq |f(n^s-1)-sf(n)|+M =$$

$$= \left|f\left((n-1)\sum_{i=0}^{s-1} n^i\right)-sf(n)\right|+M \leqq M+|f(n-1)-f(n)|+\left|f\left(\sum_{i=0}^{s-1} n^i\right)-(s-1)f(n)\right| \leqq$$

$$\leqq 2M+\left|\sum_{j=1}^{s-1}\left(f\left(\sum_{i=0}^{j} n^i\right)-f\left(\sum_{i=0}^{j-1} n^i\right)-f(n)\right)\right| \leqq$$

$$\leqq 2M+\sum_{j=1}^{s-1}\left|f\left(\sum_{i=0}^{j} n^i\right)-f\left(\sum_{i=0}^{j} n^i-1\right)\right| \leqq (s+1)M \leqq 2sM. \qquad \text{Q.E.D.}$$

Lemma 2. *For any two integers $k \geqq 0$ and $n>0$ we have*

$$|f(n^{2^k})-2^k f(n)| \leqq 4\cdot 2^k M.$$

Proof. If n is even, then the previous lemma involves this estimation with $2\cdot 2^k M$ on the right-hand side. The case if n is odd is easily derived from this latter result as follows:

$$|f(n^{2^k})-2^k f(n)| = |f((2n)^{2^k})-f(2^{2^k})-2^k f(2n)+2^k f(2)| \leqq$$
$$\leqq |f((2n)^{2^k})-2^k f(2n)|+|f(2^{2^k})-2^k f(2)| \leqq 2\cdot 2^k M+2\cdot 2^k M = 4\cdot 2^k M.$$

Lemma 3. *For any two positive integers s and t we have*

$$|f(n^t)-f(n^s)| \leqq 4|t-s|nM.$$

Proof. Suppose that e.g. $t>s$; then

$$|f(n^t)-f(n^s)| \leqq 2M+|f(n^t-1)-f(n^s-1)| \leqq$$

$$\leqq 2M+\sum_{i=s+1}^{t}|f(n^i-1)-f(n^{i-1}-1)-f(n)|+(t-s)|f(n)| =$$

$$= 2M+\sum_{i=s+1}^{t}|f(n^i-1)-f(n^i-n)|+(t-s)|f(n)| \leqq 2M+2(t-s)(n-1)M \leqq$$
$$\leqq 4(t-s)nM.$$

Here we made use of the trivial estimation

$$|f(n)| \leqq \sum_{i=1}^{n-1}|f(i+1)-f(i)| \leqq (n-1)M.$$

Thus the lemma is proved.

As an easy consequence of the above lemmas we have the existence of the limit

$$\lim_{t\to\infty} \frac{f(n^t)}{t}.$$

Indeed, this is simple if t runs only over numbers of form 2^k since by Lemma 2 we have for $k_1, k_2 \to \infty$:

$$\left|\frac{f(n^{2^{k_1}})}{2^{k_1}} - \frac{f(n^{2^{k_2}})}{2^{k_2}}\right| \leqq \frac{1}{2^{k_1}}\left|f(n^{2^{k_1}}) - \frac{f(n^{2^{k_1+k_2}})}{2^{k_2}}\right| +$$

$$+\frac{1}{2^{k_2}}\left|\frac{f(n^{2^{k_2+k_1}})}{2^{k_1}} - f(n^{2^{k_2}})\right| \leqq 4M\left(\frac{1}{2^{k_1}}+\frac{1}{2^{k_2}}\right) \to 0;$$

thus writing

$$g(n) = \lim_{k\to\infty} \frac{f(n^{2^k})}{2^k},$$

we have in particular

$$\left|\frac{f(n^{2^k})}{2^k} - g(n)\right| \leqq \frac{4M}{2^k}. \tag{1}$$

Now putting k large and fixed we have, if s runs over the primes to $n^{2^k}-1$, by Lemma 2 and Lemma 1 and by (1):

$$\left|\frac{f(n^s)}{s} - g(n)\right| \leqq$$

$$\leqq \frac{1}{s}\left|f(n^s) - \frac{1}{2^k}f(n^{s2^k})\right| + \frac{1}{2^k}\left|\frac{1}{s}f(n^{s2^k}) - f(n^{2^k})\right| + \left|\frac{1}{2^k}f(n^{2^k}) - g(n)\right| \leqq$$

$$\leqq \frac{4M}{s} + \frac{2M}{2^k} + \frac{4M}{2^k};$$

thus, for s running over the primes to $n^{2^k}-1$, we have

$$\limsup_{s\to\infty}\left|\frac{f(n^s)}{s} - g(n)\right| \leqq \frac{6M}{2^k}. \tag{2}$$

Now if $t\to\infty$ arbitrarily, $s\to\infty$ as above, and $s \leqq t < s+n^{2^k}$, then we have by Lemma 3:

$$\left|\frac{f(n^t)}{t} - \frac{s}{t}\frac{f(n^s)}{s}\right| = \frac{1}{t}\,|f(n^t)-f(n^s)| \leqq 4\frac{M}{t}(t-s)n \to 0;$$

thus if we make use of (2) and take into account that $\frac{s}{t}\to 1$ we have

$$\limsup_{t\to\infty}\left|\frac{f(n^t)}{t} - g(n)\right| \leqq \frac{6M}{2^k}.$$

The left-hand side, however, does not depend on k. So by making $k \to \infty$ we obtain

$$\limsup_{t\to\infty} \left| \frac{f(n^t)}{t} - g(n) \right| = 0, \quad \text{i. e.} \quad \lim_{t\to\infty} \frac{f(n^t)}{t} = g(n).$$

Here t tends to infinity arbitrarily.

It is obvious that here $g(n)$ is completely additive. In fact, we have

$$g(n^r) = \lim_{t\to\infty} \frac{f(n^{rt})}{t} = r \lim_{t\to\infty} \frac{f(n^{rt})}{rt} = r \lim_{t\to\infty} \frac{f(n^t)}{t} = rg(n),$$

which combined with the additivity of $g(n)$ implies complete additivity. On the other hand $h(n) = f(n) - g(n)$ is bounded; actually $|h(n)| \leqq 4M$, as inequality (1) shows with $k=0$. Thus the theorem is proved.

References

[1] P. Erdős, On the distribution function of additive function, *Ann. of Math.*, **47** (1946), 1—20.

[2] P. Erdős, On the distribution function of additive arithmetical functions and on some related problems, *Rend. Sem. Mat. Fis. Milano*, **27** (1958), 45—49.

[3] A. Máté, A new proof of a theorem of P. Erdős, *Proc. Amer. Math. Soc.*, **18** (1967), 159—162.

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(Received July 15, 1968)