

On the sum $\sum d_k(n)$

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1. Introduction

Let $d(n)$ denote the number of divisors of n , and $d_k(n)$ be the k -fold iterate of $d(n)$, i.e. $d_1(n) = d(n)$ and $d_k(n) = d(d_{k-1}(n))$ for $k \geq 2$. Let

$$(1.1) \quad D_k(x) = \sum_{n \leq x} d_k(n).$$

BELLMAN and SHAPIRO [1] conjectured that $D_k(x) = (1 + o(1)) c_k x \log_k x$ for all $k \geq 1$, where \log_k denotes the k -fold iterated logarithm.

This conjecture was proved for $k=2$ and 3 by KÁTAI [2], [3]. The aim of this paper is to prove it for $k=4$. The cases $k > 4$ seem to be essentially more difficult.

Theorem 1. *We have*

$$D_4(x) = (1 + o(1)) c x \log_4 x$$

as $x \rightarrow \infty$, where c is a positive constant.

2. Notations and decomposition of the sum $D_4(x)$

The letters $p, p_1, \dots, q, q_1, \dots$ stand for prime numbers. Let $\omega(n)$ denote the number of the different, and $\Omega(n)$ the number of all prime factors of n , i.e. for $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ let $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \dots + \alpha_r$. Let $\lambda(n) = (-1)^{\Omega(n)}$ and let $\mu(n)$ denote the Moebius function. ($|\mu(n)| = 1$ or 0 according as n is square-free or not.) Let $\sigma_a(n) = \sum_{d|n} d^a$.

The letters c, c_1, \dots denote suitable positive constants, and $\varepsilon, \varepsilon_1, \dots$ are arbitrary small positive constants not necessarily the same in every place.

We use the symbol \ll in VINOGRADOV'S sense.

For the sake of brevity denote $x_1 = \log x, x_{i+1} = \log x_i, y_1 = \log y, y_{i+1} = \log y_i$ ($i \geq 1$) and set

$$(2.1) \quad a_j(x) = \frac{(\log \log x)^{j-1}}{(j-1)!} \quad (j = 1, 2, \dots).$$

Denote by \mathcal{K} the set of integers all whose prime factors occur with an exponent greater than 1. Clearly every integer can be uniquely written in the form

$$(2.2) \quad n = Km \quad \text{with} \quad (K, m) = 1, \quad K \in \mathcal{K}, \quad m \text{ square-free.}$$

K will denote the quadratic part, m the square-free part of n . \mathcal{A}_K is the set of integers whose quadratic part is K .

For $K \in \mathcal{K}$ let the numbers k, k_1, k_2, α be defined as follows:

$$(2.3) \quad k = d(K), \quad k = 2^{\alpha} k_1, \quad k_1 \text{ odd}, \quad k_2 = d(k_1).$$

Then for an n in (2.2) we have

$$(2.4) \quad d_2(n) = (\alpha + 1 + \omega(m))k_2.$$

Set

$$(2.5) \quad \Sigma_K = \sum_{\substack{n \leq x \\ n \in \mathcal{A}_K}} d_4(n).$$

Then

$$(2.6) \quad D_4(x) = \sum_{K \in \mathcal{K}} \Sigma_K.$$

Furthermore

$$(2.7) \quad \Sigma_K = \sum_{r=1}^{\infty} \Sigma_K^r,$$

where in Σ_K^r we sum over those n for which $\omega(m) = r$ (see (2.2)). Let further

$$(2.8)-(2.9) \quad Z(y, K, r) = \sum_{\substack{(n, K)=1 \\ \omega(n)=r \\ n \leq y}} |\mu(n)|; \quad Z(y, K) = \sum_{\substack{n \leq y \\ (n, K)=1}} |\mu(n)|.$$

So by (2.4) we have

$$(2.10) \quad \Sigma_K^r = d_2(k_2(\alpha + 1 + r)) Z\left(\frac{x}{K}, K, r\right).$$

For a general natural number n let \mathcal{B}_n denote the set of those positive integers all prime factors of which occur in n . Let further

$$(2.11) \quad \tau(n) = \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} = \sum_{v \in \mathcal{B}_n} \frac{\lambda(v)}{v}.$$

Let $\pi_r(x)$ be the number of those integers not exceeding x which contain exactly r prime factors.

3. Lemmas

Lemma 1. For all $r \geq 1$ we have

$$(3.1) \quad \pi_r(x) < c_1 \frac{x(x_2 + c_2)^{r-1}}{x_1(r-1)!}.$$

This is a known theorem of HARDY and RAMANUJAN [4]. Hence we easily deduce the well-known

Lemma 2. For all constant $\delta > 0$ the inequalities

$$(3.2)-(3.3) \quad \sum_{\substack{n \leq Y \\ \Omega(n) < (1-\delta)\log_2 Y}} 1 \ll Y(\log Y)^{-\gamma_\delta}, \quad \sum_{\substack{n \leq Y \\ \Omega(n) > (1+\delta)\log_2 Y}} 1 \ll Y(\log Y)^{-\gamma_\delta}$$

hold with a suitable positive constant γ_δ . Further we have $\gamma_\delta = 2$ in (3.3).

Lemma 3. Let $h(x)$ denote an increasing function of x , tending to infinity with x . Then

$$(3.4) \quad \frac{x}{x_1} \sum_{|j-x_2| \leq h(x)\sqrt{x_2}} a_j(x) = (1 + o(1))x,$$

and consequently

$$(3.5) \quad \sum_{j \leq x_2 - h(x)\sqrt{x_2}} a_j(x) = o(x_1), \quad \sum_{j \geq x_2 + h(x)\sqrt{x_2}} a_j(x) = o(x_1).$$

Lemma 3 is well known and can be proved by a simple computation.

Lemma 4. Let $\beta < 1$ be an arbitrary positive constant, $Y_1 \cong Y_2 \cong Y_1^\beta$. Then

$$(3.6) \quad \sum_{Y_1 \cong n \leq Y_1 + Y_2} \{\omega(n) - \log_2 Y_1\}^2 \ll Y_2 \log_2 Y_1.$$

This lemma can be proved by the method of TURÁN (see [5]).

Let

$$(3.7) \quad D_h(x, t) = \sum_{\substack{x < mn^2 \leq x+h \\ n > t}} 1.$$

Lemma 5. For $0 < t \leq x^{1/3}$ and $0 < h \leq x^{2/3}$ we have

$$(3.8) \quad D_h(x, t) \ll x^{\vartheta_1} + ht^{-1} \quad \text{with } \vartheta_1 = 0,23.$$

Lemma 6. We have

$$(3.9) \quad Z(x, 1) = \frac{6}{\pi^2} x + O(x^{1/2}).$$

Furthermore, for $0 \leq h \leq x^{2/3}$,

$$(3.10) \quad Z(x+h, 1) - Z(x, 1) = \frac{6}{\pi^2} x + O(h^{1/2}) + O(x^{\vartheta_1})$$

holds.

For the proof of (3. 8) and (3. 10) see RICHERT [6]. (3. 9) is well known.

Let $I(y, c)$ denote the interval $[y_2 - c/\sqrt{y_2}, y_2 + c/\sqrt{y_2}]$. Let further A be an arbitrary but fixed constant and

$$(3. 11) \quad yy_1^{-A} \cong y^* \cong y.$$

Lemma 7. For a suitable increasing function $g(y)$ with $\lim_{y \rightarrow \infty} g(y) = \infty$ we have

$$(3. 12) \quad Z(y^*, 1, r) = \frac{6}{\pi^2} (1 + o(1)) \frac{y}{y_1} a_r(y) \quad (y \rightarrow \infty)$$

uniformly in $I(y, 4g(y))$.

This is a slightly modified form of a result of P. ERDŐS [7].

4. Further lemmas

Lemma 8. Let $b_K \ll K^\varepsilon$. Then we have

$$(4. 1) \quad \sum_{\substack{K > u \\ K \in \mathcal{X}}} \frac{b_K}{K} \ll u^{-1/3} \quad (u \rightarrow \infty).$$

Proof. This is an immediate consequence of the simple and known fact that

$$\sum_{\substack{K \leq x \\ K \in \mathcal{X}}} 1 \ll x^{1/2+\varepsilon}.$$

Lemma 9. For fixed $\beta > 0$ we have

$$(4. 2) \quad \sum_{v \in \mathcal{B}_n} v^{-\beta} \ll d(n),$$

furthermore

$$(4. 3) \quad \sum_{\substack{v \in \mathcal{B}_n \\ v > u}} v^{-\beta} \ll d(n) u^{-\gamma}$$

when $\gamma < \beta$ and γ is constant.

Proof. Since

$$\sum_{v \in \mathcal{B}_n} v^{-\beta} = \prod_{p|n} \left(1 - \frac{1}{p^\beta}\right)^{-1} = \prod_{p^\beta < 2} \cdot \prod_{p^\beta \geq 2} \cong C(\beta) d(n)$$

which proves (4. 2). Now (4. 2) implies (4. 3) since $v^{-\beta} \cong u^{-\gamma} v^{-\beta+\gamma}$ for $v \geq u$.

Lemma 10. We have

$$(4. 4) \quad Z(y, K, r) = \frac{6}{\pi^2} (1 + o(1)) \tau(K) \frac{y}{y_1} a_r(y) \quad (y \rightarrow \infty)$$

uniformly for $K \cong y_2^4$, $r \in I(y, 2g(y))$. [$g(y)$ as in Lemma 7.]

Proof. The identity

$$(4.5) \quad Z(y, K, r) = \sum_{\substack{v \in \mathcal{B}_K \\ v \leq \Delta}} \lambda(v) Z\left(\frac{y}{v}, 1, r - \Omega(v)\right)$$

can be proved elementarily or by using the uniqueness of Dirichlet series expansions.

Suppose that $K \cong y_2^4, r \in I(y, 2g(y))$. Let $\Delta = y_2^2$. For $v < \Delta$ we have $\Omega(v) < > c \log 2v < cy_3 \cong g(y)y_2^{1/2}$. Hence $r - \Omega(v) \in I(y, 2g(y))$, if y is sufficiently large. From (4.5) we obtain

$$(4.6) \quad Z(y, K, r) = \sum_{\substack{v \in \mathcal{B}_K \\ v \leq \Delta}} \lambda(v) Z\left(\frac{y}{v}, 1, r - \Omega(v)\right) + \\ + O\left(\sum_{\substack{v \in \mathcal{B}_K \\ v > \Delta}} Z\left(\frac{y}{v}, 1, r - \Omega(v)\right)\right) = \Sigma_1 + O(\Sigma_2).$$

Using Lemma 7 we deduce

$$(4.7) \quad \Sigma_1 = \frac{6}{\pi^2} (1 + o(1)) \frac{y}{y_1} \sum_{\substack{v \in \mathcal{B}_K \\ v \leq \Delta}} \frac{\lambda(v)}{v} a_{r - \Omega(v)}(y).$$

Since $a_{r - \Omega(v)}(y) = (1 + o(1))a_r(y)$ in $r \in I(y, 2g(y))$ we have

$$\Sigma_1 = \frac{6}{\pi^2} (1 + o(1)) \tau(K) \frac{ya_r(y)}{y_1} + o(1)a_r(y) \frac{y}{y_1} \sum_{v \in \mathcal{B}_K} 1_v + O\left(\frac{y}{y_1} a_r(y) \sum_{\substack{v \in \mathcal{B}_K \\ v > \Delta}} \frac{1}{v}\right).$$

Hence by $\sum_{v \in \mathcal{B}_K} v^{-1} \ll \tau(K)$ and (4.3) we obtain

$$\Sigma_1 = \frac{6}{\pi^2} (1 + o(1)) \tau(K) \frac{ya_r(y)}{y_1}.$$

Now we estimate the sum Σ_2 . We have by (4.2) that

$$\Sigma_2 \cong y \sum_{v > \Delta} v^{-1} \ll yd(K)\Delta^{-1/2} \ll yy_2^{-2}$$

and so $\Sigma_2 = o(\Sigma_1)$. Hence (4.4) follows.

Lemma 11. We have

$$(4.8) \quad Z(y, K) = \frac{6}{\pi^2} \tau(K)y + O(d(K)y^{1/2}).$$

Proof. Summing in (4.5) for $1 \leq r < \infty$ we deduce

$$(4.9) \quad Z(y, K) = \sum_{\substack{v \leq y \\ v \in \mathcal{B}_K}} \lambda(v) Z\left(\frac{y}{v}, 1\right).$$

Hence by (3. 9) we have

$$Z(y, K) = \frac{6}{\pi^2} \tau(K)y + O\left(y \sum_{\substack{v \geq y \\ v \in \mathcal{B}_K}} \frac{1}{v}\right) + O\left(y^{1/2} \sum_{v \in \mathcal{B}_K} \frac{1}{\sqrt{v}}\right).$$

By Lemma 9 we deduce (4. 8).

Lemma 12. *Let $z_1^{2/3} \cong z_2 \cong z_1^{1/4}$, $L = O(z_1^{1/4})$. Then we have*

$$(4. 10) \quad Z(z_1 + z_2, L) - Z(z_1, L) = \frac{6}{\pi^2} \tau(L)z_2 + O(d(L)(z_1^{1/4} + z_2^{1/2})).$$

Proof. Using the identity (4. 9) we have

$$\Delta \stackrel{\text{def}}{=} Z(z_1 + z_2, L) - Z(z_1, L) = \sum_{\substack{v \in \mathcal{B}_L \\ v < z_1 + z_2}} \lambda(v) \left\{ Z\left(\frac{z_1 + z_2}{v}, 1\right) - Z\left(\frac{z_1}{v}, 1\right) \right\}.$$

Hence by (3. 10) we obtain

$$\begin{aligned} \Delta &= \frac{6}{\pi^2} \tau(L)z_2 + O\left(z_2 \sum_{\substack{v > z_2 \\ v \in \mathcal{B}_L}} \frac{1}{v}\right) + O(z_1^{1/4} \sum_{v \in \mathcal{B}_L} v^{-1}) + \\ &\quad + O(\sqrt{z_2} \sum_{v \in \mathcal{B}_L} v^{-1/2}) + O\left(\sum_{\substack{z_2 < v < z_1 + z_2 \\ v \in \mathcal{B}_L}} 1\right). \end{aligned}$$

For the last sum we have

$$\sum_{\substack{z_2 < v < z_1 + z_2 \\ v \in \mathcal{B}_L}} 1 < (2z_1)^{0,1} \sum_{v \in \mathcal{B}_L} v^{-0,1} \ll d(L)z_1^{0,1}.$$

Using Lemma 9 for the other remainder terms we have (4. 10).

5. For a general integer S let

$$(5. 1) \quad T_S(Y_1, Y_1 + Y_2) = \sum_{Y_1 < r \cong Y_1 + Y_2} d_2(Sr).$$

Every integer r can be represented in the form

$$(5. 2) \quad r = R_1 R_2 \varrho, \quad R_1 \in \mathcal{B}_S, (R_2 \varrho, S) = 1, \quad R_2 \in \mathcal{K}, \quad |\mu(\varrho)| = 1$$

and this representation is unique.

Let $L = R_1 R_2$ and D_L be the set of those r in (5. 2) for which $L = R_1 R_2$. Let

$$(5. 3) \quad d(SL) = l = 2^\beta l_1, \quad \text{with } l_1 \text{ odd and } d(l_1) = l_2,$$

and let

$$(5. 4) \quad A(S) = \sum_{R_1, R_2} \frac{l_2 \tau(SR_2)}{R_1 R_2}.$$

Lemma 13. Let $Y_1^{1/2} \cong Y_2 \cong Y_1^{1/3}$, $S \cong Y_1^{0.01}$. Then

$$(5.5) \quad T_S(Y_1, Y_1 + Y_2) = \frac{6}{\pi^2} A(S) Y_2 \log \log Y_1 + O(Y_2 (\log_2 Y_1)^{1/2} S^\epsilon).$$

Proof. Using the notations in (5.3) we have for an r in (5.2)

$$(5.6) \quad d_2(Sr) = (\omega(r) + \beta + 1)l_2 = \omega(r)l_2 + (\beta + 1 - \omega(L))l_2.$$

Hence

$$(5.7) \quad T_S(Y_1, Y_1 + Y_2) = \sum_{Y_1 < r \cong Y_1 + Y_2} \omega(r)l_2 + \sum_{Y_1 < r \cong Y_1 + Y_2} (\beta + 1 - \omega(L))l_2 = \Sigma_1 + \Sigma_2.$$

Furthermore

$$(5.8) \quad \begin{aligned} \Sigma_1 &= \log_2 Y_1 \sum_{Y_1 < r \cong Y_1 + Y_2} l_2 + O\left(\sum_{Y_1 < r \cong Y_1 + Y_2} |\omega(r) - \log_2 Y_1| l_2\right) = \\ &= \log_2 Y_1 \Sigma_3 + O(\Sigma_4). \end{aligned}$$

By the Cauchy inequality we have

$$(5.9) \quad \Sigma_4 \cong \left\{ \sum_{Y_1 < r \cong Y_1 + Y_2} (\omega(r) - \log_2 Y_1)^2 \right\}^{1/2} \left\{ \sum_{Y_1 < r \cong Y_1 + Y_2} l_2^2 \right\}^{1/2} = \Sigma_5^{1/2} \Sigma_6^{1/2}.$$

By Lemma 4

$$(5.10) \quad \Sigma_5 \ll Y_2 \log_2 Y_1.$$

Using (5.3) we obtain $(d(m) < m^\epsilon)$

$$l_2^2 = O((SL)^\epsilon) (\beta + 1 - \omega(L))l_2 = O((SL)^\epsilon).$$

Consequently,

$$(5.11) \quad \Sigma_2 = O(S^\epsilon \Sigma_7), \quad \Sigma_6 = O(S^\epsilon \Sigma_7),$$

where

$$(5.12) \quad \Sigma_7 = \sum_{Y_1 < r \cong Y_1 + Y_2} L^\epsilon.$$

We have

$$(5.13) \quad \Sigma_7 \ll \sum_{\substack{Y_1 < r = qL \cong Y_1 + Y_2 \\ L \cong Y_2}} L^\epsilon + Y_1^\epsilon \sum_{\substack{Y_1 < r = qL \cong Y_1 + Y_2 \\ L > Y_2}} 1 = \Sigma_8 + Y_1^\epsilon \Sigma_9.$$

Furthermore

$$(5.14) \quad \begin{aligned} \Sigma_8 &\ll Y_2 \sum_{R_1 R_2 \cong Y_2} (R_1 R_2)^{\epsilon-1} \ll Y_2 \left\{ \sum_{R_1 \in \mathcal{B}_S} R_1^{\epsilon-1} \right\} \left\{ \sum_{R_2 \in \mathcal{A}} R_2^{\epsilon-1} \right\} \ll \\ &\ll Y_2 \prod_{p|S} \left(1 - \frac{1}{p^{1-\epsilon}} \right)^{-1} \ll d(S) Y_2 \ll Y_2 S^\epsilon. \end{aligned}$$

Now we estimate the sum Σ_9 .

Let u^2 and v^2 denote the greatest square divisors of the numbers R_1 and R_2 . Since $R_1 \in \mathcal{B}_S$, so $u^2 \cong R_1/S \cong R_1 Y_1^{-0.01} \cong R_1 Y_2^{0.03}$ holds. Furthermore, since all prime factors of R_2 occur with an exponent greater than 1, we have $R_2 = v^2 l$

and l/v , i.e. $v \cong R_2^{1/3}$. Hence the in $r\Sigma_9$ have the form $r = n^2m$, where $n \cong R_1 R_2^{1/3} Y_2^{-0,03} \cong (R_1 R_2)^{1/3} Y_2^{-0,03} \cong Y_2^{0,3}$. Thus

$$\Sigma_9 \ll \sum_{\substack{Y_1 \cong n^2 m \cong Y_1 + Y_2 \\ n \cong Y_2^{0,3}}} 1.$$

Applying Lemma 5 we obtain

$$(5.15) \quad \Sigma_9 \ll Y_1^{1/4} + Y_2^{0,7} \ll Y_2^{3/4}.$$

Combining this with (5.14) we deduce

$$(5.16) \quad T_S(Y_1, Y_1 + Y_2) = \log_2 Y_1 \cdot \Sigma_3 + O(Y_2 (\log_2 Y_1)^{1/2} S^\epsilon).$$

Now we estimate the sum Σ_3 . We have by (5.15)

$$(5.17) \quad \Sigma_3 = \sum_{\substack{Y_1 \cong r \cong Y_1 + Y_2 \\ L \cong Y_2}} I_2 + O(Y_1^\epsilon \sum_{\substack{Y_1 < r \cong Y_1 + Y_2 \\ L > Y_2}} 1) = \Sigma_{10} + O(Y_1^\epsilon \Sigma_9) = \Sigma_{10} + O(Y_2^{0,8}).$$

Furthermore by (5.2)

$$(5.18) \quad \begin{aligned} \Sigma_{10} &= \sum_{L \cong Y_2} I_2 \left\{ Z \left(\frac{Y_1 + Y_2}{L}, SL \right) - Z \left(\frac{Y_1}{L}, SL \right) \right\} = \\ &= \sum_{L \cong Y_2^{0,01}} + \sum_{L > Y_2^{0,01}} = \Sigma_{11} + \Sigma_{12}. \end{aligned}$$

For Σ_{12} we have

$$(5.19) \quad \begin{aligned} \Sigma_{12} \ll Y_2 \sum_{L \cong Y_2^{0,01}} \frac{I_2}{L} \ll Y_2 \sum_{L \cong Y_2^{0,01}} L^{-1+\epsilon} \ll Y_2 \left\{ \sum_{R_1} R_1^{-1+\epsilon} \right\} \left\{ \sum_{R_2 > Y_2^{0,005}} R_2^{-1+\epsilon} \right\} + \\ + Y_2 \left\{ \sum_{R_1 > Y_2^{0,005}} R_1^{-1+\epsilon} \right\} \left\{ \sum_{R_2} R_2^{-1+\epsilon} \right\} \ll Y_2^{1-0,0001}. \end{aligned}$$

For Σ_{11} we use Lemma 12 and deduce

$$(5.20) \quad \begin{aligned} \Sigma_{11} &= \frac{6}{\pi^2} Y_2 \sum_{L \cong Y_2^{0,01}} \frac{I_2 \tau(LS)}{L} + O \left(Y_1^{1/4} \sum_{L \cong Y_2^{0,01}} \frac{d(L) I_2}{L^{1/4}} \right) + \\ &+ O \left(Y_2^{1/2} \sum_{L \cong Y_2^{0,01}} \frac{d(L) I_2}{L^{1/2}} \right) = \frac{6}{\pi^2} Y_2 \Lambda(S) + O \left(\sum_{L \cong Y_2^{0,01}} \frac{I_2 \tau(LS)}{L} \right) + O(Y_2^{3/4}). \end{aligned}$$

Further, by elementary calculation,

$$(5.21) \quad \sum_{L \cong Y_2^{0,01}} I_2 \frac{\tau(LS)}{L} \cong \frac{\tau(S)}{(Y_2^{0,01})^{1/4}} Y_2 \sum_L \frac{\tau(L)}{L^{3/4}} \ll Y_2^{-0,001}.$$

Combining our inequalities (5.17)–(5.21), (5.5) follows and hence the lemma is proved.

Putting $S=1$ in Lemma 13 we obtain by a simple calculation

$$D_2(x) = \sum_{n \cong x} d_2(n) = cx \log_2 x + O(x \sqrt{\log_2 x}).$$

6. The proof of the theorem

First we prove that

$$(6.1) \quad \Sigma_1 \stackrel{\text{def}}{=} \sum_{\substack{K > x_2^3 \\ K \in \mathcal{K}}} \Sigma_K \ll x.$$

Indeed by (2.3), (2.4) we have

$$d_4(n) \leq d_2(n) = (\alpha + 1 + \omega(m))k_2, \quad \alpha \ll \log K.$$

Let $\Sigma_1 = \Sigma_2 + \Sigma_3$, with $\omega(m) \leq 10x_2$ in Σ_2 and $\omega(m) > 10x_2$ in Σ_3 . So by Lemma 8 we have

$$\Sigma_2 \ll x \sum_{\substack{K > x_2^3 \\ K \in \mathcal{K}}} \frac{(\log K + x_2)k_2}{K} \ll x.$$

Furthermore, using that $\omega(m) \ll x_1$, we have

$$\Sigma_3 \ll x_1 \sum_{\omega(n) \leq 10x_2} k_2 \ll x_1 \left\{ \sum_{n \leq x} k_2^2 \right\}^{1/2} \left\{ \sum_{\substack{\omega(n) \leq 10x_2 \\ n \leq x}} 1 \right\}^{1/2} \ll xx_1 \left\{ \sum \frac{k_2^2}{K} \right\}^{1/2} x_1^{-1} \ll x,$$

by Lemma 8 and Lemma 2.

Suppose now that $K \leq x_2^3$. Let

$$(6.2)-(6.3) \quad \Sigma_K^{(-)} = \sum_{r \leq \frac{1}{2}x_2} \Sigma_K^r, \quad \Sigma_K^{(+)} = \sum_{r \geq 2x_2} \Sigma_K^r,$$

$$(6.4) \quad \Sigma_K^{(0)} = \sum_{\frac{1}{2}x_2 < r < 2x_2} \Sigma_K^r.$$

We prove that

$$(6.5) \quad \Sigma^{(-)} = \sum_{\substack{K \leq x_2^3 \\ K \in \mathcal{K}}} \Sigma_K^{(-)} = O(x),$$

and that

$$(6.6) \quad \Sigma^{(+)} = \sum_{\substack{K \leq x_2^3 \\ K \in \mathcal{K}}} \Sigma_K^{(+)} = O(x).$$

Since $K \leq x_2^3$ we have $\omega(K) \ll x_3$ and so in the sums $\Sigma_K^{(-)} \omega(n) \ll \frac{3}{4}x_2$. Furthermore we have $d_4(n) \leq G(\varepsilon)d^\varepsilon(n)$. So by the Hölder inequality

$$\Sigma^{(-)} \ll \sum_{\substack{n \leq x \\ \omega(n) \leq \frac{3}{4}x_2}} d^\varepsilon(n) \ll \left\{ \sum_{\omega(n) \leq \frac{3}{4}x_2} 1 \right\}^{1-\varepsilon} \left\{ \sum d(n) \right\}^\varepsilon \ll x \cdot x_1^{\varepsilon - \gamma_1/(1-\varepsilon)} \ll x,$$

if ε is small enough (see (3.2)). The proof of (6.6) is very similar.

Finally we prove that

$$(6.7) \quad \Sigma^{(0)} \stackrel{\text{def}}{=} \sum_{K < x_2^3} \Sigma_K^{(0)} = c(1 + o(1)) \cdot xx_4.$$

Since $D_4(x) = \Sigma^{(0)} + \Sigma^{(+)} + \Sigma^{(-)} + \Sigma_1$, the theorem will immediately follow.

By (2. 10) we have

$$(6.8) \quad \Sigma_K^{(0)} = \sum_{\frac{x_2}{2} < r < 2x_2} d_2(k_2(\alpha + 1 + r)) Z \left(\frac{x}{K}, K, r \right) = \Sigma_K^{(1)} + \Sigma_K^{(2)},$$

where in $\Sigma_K^{(1)}$ $|r - x_2| \leq g(x)\sqrt{x_2}$ and in $\Sigma_K^{(2)}$ $|r - x_2| \geq g(x)\sqrt{x_2}$ holds. Here $g(x)$ is a sequence which tends to infinity with α monotonically and for which the Lemma 10 holds.

Let $A = [x_2^{1/3}]$, $A_x = x_2 - g(x)\sqrt{x_2}$, $B_x = x_2 + g(x)\sqrt{x_2}$ and split the interval into consecutive subintervals with lengths A . Let

$$G_j = [A_x + (j-1)A, A_x + jA], \quad j = 1, 2, \dots, T; \quad T = \left\lfloor \frac{2g(x)\sqrt{x_2}}{A} \right\rfloor + 1.$$

Thus we have by (4. 4) that

$$(6.9) \quad \Sigma_K^{(1)} = \sum_{j=1}^T \Sigma_K^{(1,j)} + O(\Sigma_K^{(1,T)}),$$

where

$$\Sigma_K^{(1,j)} = \sum_{r \in G_j} d_2(k_2(\alpha + 1 + r)) Z \left(\frac{x}{K}, K, r \right).$$

By Lemma 10 we have

$$\Sigma_K^{(1,j)} = \frac{6}{\pi^2} (1 + o(1)) \frac{\tau(K)}{K} \frac{x}{\log x/K} \sum_{r \in G_j} d_2(k_2(\alpha + 1 + r)) a_r \left(\frac{x}{K} \right).$$

Taking into account that $a_r(x/K) = (1 + o(1))a_r(x)$ for $K \leq x_2^4$, $r \in I(x, 2g(x))$ and that $a_{r_1}(x)/a_{r_2}(x) = 1 + o(1)$ for $|r_1 - r_2| \leq A$, $r_1, r_2 \in I(x, 2g(x))$ we have

$$\begin{aligned} & \Sigma_K^{(1,j)} = \\ & = \frac{6}{\pi^2} (1 + o(1)) \frac{\tau(K)}{K} \frac{x}{x_1} T_{k_2}(A_x + (j-1)A + \alpha + 1, A_x + jA + \alpha + 1) \frac{1}{A} \sum_{r \in G_j} a_r(x). \end{aligned}$$

Observing that the conditions of Lemma 13 are satisfied, we deduce

$$\Sigma_K^{(1,j)} = \left(\frac{6}{\pi^2} \right)^2 (1 + o(1)) \frac{\tau(K)}{K} \Lambda(k_2) \frac{x}{x_1} x_4 \sum_{r \in G_j} a_r(x) + O \left(\frac{\tau(K)}{K} k_2^2 \frac{x}{x_1} x_4^{1/2} \sum_{r \in G_j} a_r(x) \right).$$

Hence by (6. 9) using (3. 4) we have

$$(6.10) \quad \Sigma_K^{(1)} = (1 + o(1)) \left(\frac{6}{\pi^2} \right)^2 \frac{\tau(K)}{K} \Lambda(k_2) xx_4 + O \left(xx_4^{1/2} \frac{\tau(K)}{K} k_2^2 \right).$$

Now we consider the sum $\Sigma_K^{(2)}$. From (3. 1) we easily deduce

$$Z\left(\frac{x}{K}, K, r\right) < c_1 \frac{x}{Kx_1} \frac{(x_2 + c_2)^{r-1}}{(r-1)!} < c \frac{x}{Kx_1} a_r(x)$$

for $r < 2x_2$ and $K < x_2^3$. Hence we have

$$(6. 11) \quad \Sigma_K^{(2)} \ll \frac{x}{Kx_1} \left\{ \sum_{\frac{x_2}{2} \leq r \leq Ax} d_2(k_2(\alpha + 1 + r)) a_r(x) + \sum_{B_x \leq r \leq 2x_2} d_2(k_2(\alpha + 1 + r)) a_r(x) \right\}.$$

Let Σ_4 and Σ_5 denote the first and the second sum in the right hand side of (6. 11). Taking into account that $a_r(x)$ is monotonically increasing in Σ_4 and decreasing in Σ_5 in r , we have

$$\Sigma_4 \cong \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x) \sum_{A_x - jA \leq r \leq A_x + (j+1)A} d_2(k_2(\alpha + 1 + r))$$

and similarly

$$\Sigma_5 \cong \sum_{j=1}^{\left[\frac{x_2}{2A}\right]} a_{B_x + jA}(x) \sum_{B_x + (j-1)A \leq r \leq B_x + jA} d_2(k_2(\alpha + 1 + r)).$$

Hence by Lemma 13 we have

$$\Sigma_4 \ll \{x_4 \Lambda(k_2) + O(x_4^{1/2} k_2^{\epsilon})\} A \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x).$$

Since

$$A \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x) \cong \sum_{r \leq A_x + A} a_r(x) = o(x_1),$$

we have

$$(6. 12) \quad \Sigma_4 = o(x_1) \{x_4 \Lambda(k_2) + x_4^{1/2} k_2^{1/2}\}.$$

Using similar arguments we can deduce for Σ_5 the same inequality.

Hence by (6. 11) and (6. 10)

$$\Sigma_K^{(2)} = o(1) \Sigma_K^{(1)} \quad \text{i.e.} \quad \Sigma_K^{(0)} = (1 + o(1)) \Sigma_K^{(1)}.$$

Summing over K we have

$$\Sigma^{(0)} = (1 + o(1)) \left(\frac{6}{\pi^2}\right)^2 x x_4 \sum_{K \leq x_2^3} \frac{\tau(K)}{K} \Lambda(k_2) + O\left(x x_4^{1/2} \sum_{K \leq x_2^3} \frac{\tau(K) k_2^{\epsilon}}{K}\right).$$

Observing that the sums are convergent we deduce (6. 7).

This completes the proof of our theorem.

References

- [1] R. BELLMAN and H. N. SHAPIRO, On a problem in additive number theory, *Ann. of Math.*, **49** (1948), 333—340.
- [2] I. KÁTAI, On the sum $\Sigma dd(f(n))$, *Acta Sci. Math.*, **29** (1968), 199—205.
- [3] I. KÁTAI, On the iteration of the divisor function, *Publ. Math. Debrecen* (in print).
- [4] G. H. HARDY and S. RAMANUJAN, The normal number of prime factors of a number n , *Quart. J. Pure and Appl. Math.*, **48** (1920), 76—92.
- [5] P. TURÁN, Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan, *J. London Math. Soc.*, **11** (1936), 125—133.
- [6] H. E. RICHERT, On the difference between consecutive square free numbers, *J. London Math. Soc.*, **29** (1954), 16—20.
- [7] P. ERDŐS, On the integers having k prime factors, *Ann. of Math.*, **49** (1948), 53—66.

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